1. Algebra of the dot and cross product

The dot product can be formed with any two n-vectors: the cross product can only be formed with two 3-vectors so whenever we write a formula with a cross product it is assumed that the vectors are 3-vectors.

(1) For vectors $\vec{A}$ and $\vec{B}$ and scalar c:
$(c\vec{A}) \bullet \vec{B} = \vec{A} \bullet (c\vec{B}) = c(\vec{A} \bullet \vec{B})$ and $(c\vec{A}) \times \vec{B} = \vec{A} \times (c\vec{B}) = c(\vec{A} \times \vec{B})$.

(2) For vectors $\vec{A}$ and $\vec{B}$, $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$ and $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

(3) For vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$:
$(\vec{A} + \vec{B}) \bullet \vec{C} = \vec{A} \bullet \vec{C} + \vec{B} \bullet \vec{C}$; $(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$ and
$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$; $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.

(4) If θ is the angle between the two vectors $\vec{A}$ and $\vec{B}$ with $0 \leqslant \theta \leqslant \pi$, then
$\vec{A} \bullet \vec{B} = |\vec{A}| \cdot |\vec{B}| \ \cos(\theta)$ and $|\vec{A} \times \vec{B}| = |\vec{A}| \cdot |\vec{B}| \ \sin(\theta)$.

(5) For vectors, $\vec{A}$, $\vec{B}$ and $\vec{C}$,
$\vec{A} \bullet (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \bullet \vec{C}$: $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$. $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{B} \bullet \vec{C})\vec{A}$.

Neither product has an associative law.

2. Cross product

2.1. Determinants.

$$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c$$

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot \det \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \det \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \det \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det \begin{vmatrix} \vdots & \vdots & \vdots \\ \cdots & a_{row\ column} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix} = a_{11} \cdot \det |A_{11}| - a_{12} \cdot \det |A_{12}| + \cdots + (-1)^{i+1} a_{1i} \cdot \det |A_{1i}| + \cdots$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by crossing out the i^{th} row and the j^{th} column from the original matrix.

2.2. Cross product.

If $\vec{A} = (a_1, a_2, a_3)$ and $\vec{B} = (b_1, b_2, b_3)$ then

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Given $n-1$ n-vectors, a similar formula can be used to produce a new vector.

2.2.1. *Additional properties of the cross product.*

(1) $\vec{A} \times \vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$.

(2) $\vec{A} \times \vec{B} = \vec{0}$ if and only if $\vec{A}$ and $\vec{B}$ are parallel.

(3) $|\vec{A} \times \vec{B}|$ is the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$ in the plane spanned by $\vec{A}$ and $\vec{B}$.

(4) $|\vec{A} \bullet (\vec{B} \times \vec{C})|$ is the volume of the parallelepiped determined by the three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$.

(5) $|\vec{A} \bullet (\vec{B} \times \vec{C})| = \det \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$.

3. Projections

(1) Project vector $\vec{A}$ onto vector $\vec{B}$: $\text{proj}_{\vec{B}}(\vec{A}) = \left(\frac{\vec{A}\bullet\vec{B}}{\vec{B}\bullet\vec{B}}\right)\vec{B}$.
(2) Project vector $\vec{A}$ into the plane P, $\vec{N}$. Note the point is irrelevant. Project $\vec{A}$ onto $\vec{N}$, $\text{proj}_{\vec{N}}(\vec{A}) = \left(\frac{\vec{A}\bullet\vec{N}}{\vec{N}\bullet\vec{N}}\right)\vec{N}$ and then take $\vec{A} - \text{proj}_{\vec{N}}(\vec{A})$.

4. Distances

(1) Point to point: $\vec{P} = P_1 - P_0$ and then the distance is $|\vec{P}|$, the square root of the sum of the squares of the coordinates.
(2) Point to line: $P_0 = (a_0, b_0, c_0)$ to the line $P_1 = (a_1, b_1, c_1)$, $\vec{V}$. Project $P = P_1 - P_0$ onto $\vec{V}$ and compute the norm of $\vec{V} - \text{proj}_{\vec{V}}(\vec{P})$.
(3) Point to plane: $P_0 = (a_0, b_0, c_0)$ to the plane $P_1 = (a_1, b_1, c_1)$, $\vec{N}$. Project $P = P_1 - P_0$ onto $\vec{N}$ and compute the norm.
(4) Line to plane: line P_0, $\vec{V}$ and plane P_1, $\vec{N}$. If $\vec{V} \bullet \vec{N} \neq 0$, the line intersects the plane and the distance is 0. If $\vec{V} \bullet \vec{N} = 0$, the line and the plane are parallel so compute the distance from P_0 to the plane.
(5) Plane to plane: plane P_0, $\vec{N}_0$ and plane P_1, $\vec{N}_1$. Compute $\vec{N}_0 \times \vec{N}_1$: if it is $\vec{0}$ the planes are parallel so compute the distance from P_0 to the plane P_1, $\vec{N}_1$; if it is non-zero the planes intersect and the answer is 0.
(6) Line to line: line P_0, $\vec{V}_0$ and plane P_1, $\vec{V}_1$. Compute $\vec{N} = \vec{V}_0 \times \vec{V}_1$. If $\vec{N} = \vec{0}$ the two lines are parallel: compute the distance from P_0 to the line P_1, $\vec{V}_1$. If $\vec{N} \neq \vec{0}$, the first line lies in the plane P_0, $\vec{N}$ and the second line lies in the parallel plane P_1, $\vec{N}$. Compute the distance from P_0 to the plane P_1, $\vec{N}$.

5. Intersections

(1) Point and point: intersection is either the point or is empty. Eyeball it!
(2) Point and line: P_0 and line P_1, $\vec{V}$. Intersection is empty or P_0. It is P_0 if and only if $\vec{P} = P_1 - P_0$ is parallel to $\vec{V}$. Compute a cross product or eyeball it.
(3) Point and plane: P_0 and plane P_1, $\vec{N}$. Intersection is empty or P_0. It is P_0 if and only if $(P_1 - P_0) \bullet \vec{V} = 0$.
(4) Line and plane: line P_0, $\vec{V}$ and plane P_1, $\vec{N}$. Intersection is either a point or the line is parallel to the plane in which case the intersection is either empty or the entire line. Compute $\vec{V} \bullet \vec{N}$. If it is 0 the line and the plane are parallel. If P_0 lies in the plane, the entire line is in the plane, otherwise the intersection is empty. If $\vec{V} \bullet \vec{N} \neq 0$ find the unique t such that the point $P_0 + t\vec{V}$ is in the plane P_1, $\vec{N}$. See (3).
(5) Plane and plane: plane P_0, $\vec{N}_0$ and P_1, $\vec{N}_1$. Intersection is a line, or the two planes are parallel and the intersection is empty or both planes are the same. Compute $\vec{V} = \vec{N}_0 \times \vec{N}_1$. If $\vec{V} = \vec{0}$ the two planes are parallel. If P_0 lies in P_1, $\vec{N}_1$, the two planes are identical, otherwise they do not intersect: see (3). If $\vec{V} \neq \vec{0}$, then there is a line of intersection and the vector for this line is $\vec{V}$. To find a point on the line, find a point in both planes. Using (3), let $\vec{X} = (x, y, z)$ and find one solution to the two equations $\vec{X} \bullet \vec{N}_0 = P_0 \bullet \vec{N}_0$ and $\vec{X} \bullet \vec{N}_1 = P_1 \bullet \vec{N}_1$.

(6) Line and line: line P_0, $\vec{V}_0$ and line P_1, $\vec{V}_1$. Intersection is either a point, empty or both lines are the same. Solve $t\vec{V}_0 + P_0 = s\vec{V}_1 + P_1$.

6. One more problem

Prof. Banchoff pointed out after class that there was no discussion of why the distance formula for a line to a line works. This point was addressed in example 10, §13.5, p. 837 of the book. The issue is that by definition the distance between the two lines is the shortest distance between two points, one on each line. Hence example 10 started by finding the points on each line closest to one another.

This can be done as follows. The lines are P_0, $\vec{V}_0$ and P_1, $\vec{V}_1$. The vector $\vec{N} = \vec{V}_0 \times \vec{V}_1$ is non-zero so the two planes P_0, $\vec{N}$ and P_1, $\vec{N}$ are parallel and each plane contains one of the lines.

Let $\vec{X}_0 = (x, y, z)$ be a point in the first plane, so $\vec{X}_0 \bullet \vec{N} = P_0 \bullet \vec{N}$. We want to compute the projection of the vector $\vec{X}_0 - P_1$ into the plane P_1, $\vec{N}$. It is

$$\vec{X}_0 - P_1 - \frac{(\vec{X}_0 - P_1) \bullet \vec{N}}{\vec{N} \bullet \vec{N}} \vec{N}$$

Hence the projection of the line P_0, $\vec{V}_0$ onto the plane P_1, $\vec{N}$ is

$$t\vec{V}_0 + P_0 - P_1 - \frac{(t\vec{V}_0 + P_0 - P_1) \bullet \vec{N}}{\vec{N} \bullet \vec{N}} \vec{N}$$

or, if $\vec{P} = P_0 - P_1$

$$t\vec{V}_0 + \vec{P} - \frac{\vec{P} \bullet \vec{N}}{\vec{N} \bullet \vec{N}} \vec{N} = t\vec{V}_0 + P_0'$$

Now the lines P_0', $\vec{V}_0$ and P_1, $\vec{V}_1$ are in the same plane and not parallel so they intersect in one point, which can be found using (6) of the Intersections section. The intersection point is given by the unique (t, s) such that $t\vec{V}_0 + P_0' = s\vec{V}_1 + P_1$.

The point $t\vec{V}_0 + P_0'$ is the projection of the point $t\vec{V}_0 + P_0$ so the distance between the two points $t\vec{V}_0 + P_0$ and $s\vec{V}_1 + P_1$ is the distance between the two planes. Hence the distance between the two lines is at most the distance between the two planes. On the other hand this is the smallest the distance can be.

In other words, the formula for the distance from a plane to a plane given in (6) works without actually having to find the two points on the line which are actually closest. If you want to solve this harder problem, then this section shows you how.