Scratch Sols for Practice Exam

1/ $\angle ABC$ = angle between
$\overrightarrow{AB} = \langle -3, 4, 0 \rangle$ and $\overrightarrow{AC} = \langle 1, 3, -2 \rangle$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = -3 \cdot 1 + 4 \cdot 3 + 0 \cdot -2 = 9 = |\overrightarrow{AB}| \cdot |\overrightarrow{AC}| \cos\theta$$

$|\overrightarrow{AB}| = 5 \quad |\overrightarrow{AC}| = \sqrt{14}$

$$\frac{9}{5 \cdot \sqrt{14}} = \cos\theta$$

$\cancel{.481 = \cos\theta}$

$\cancel{\cos\theta \approx 1.0}$

$$\theta = \cos^{-1}\left(\frac{9}{5\sqrt{14}}\right)$$

2/ $$\text{comp}_a b = \frac{a \cdot b}{|a|} = \frac{\langle -1, 4, 8 \rangle \cdot \langle 2, -3, 6 \rangle}{|\langle -1, 4, 8 \rangle|}$$

$$= \frac{-2 - 1\overset{2}{\cancel{6}} + 6 \cdot 8}{\sqrt{1^2 + 16 + 8^2}} = \frac{34}{9}$$

3/ distance given by

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$(-1, 0, -2) = (x_0, y_0, z_0)$

$a = 1$

$b = -2$

$c = 3$

$d = -7$

Since $ax + by + cz + d = 0$
eq of plane

distance =

$$\frac{|1 \cdot -1 + \cancel{(1)} -2 \cdot 0 + (3 \cdot -2) - 7|}{\sqrt{1^2 + 2^2 + 3^2}}$$

$$= \frac{14}{\sqrt{14}} = \frac{14}{\sqrt{14}} \cdot \frac{\sqrt{14}}{\sqrt{14}} = \boxed{\sqrt{14}}$$

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d 4/

$$r(t) = \int \langle 2t, -\sin(t), e^t \rangle \, dt$$

$$= \langle t^2, \cos(t), e^t \rangle + \vec{c}$$

$$\langle 2, -2, 2 \rangle = r(0) = \langle 0^2, \cos(0), e^0 \rangle + \vec{c}$$

$$= \langle 0, 1, 1 \rangle + \vec{c}$$

$$\langle 2, -3, 1 \rangle = \vec{c}$$

$$r(t) = \langle t^2 + 2, \cos(t) - 3, e^t + 1 \rangle$$

$r(t)=\langle 2t, \cos(t), 2\sin(t)\rangle$

5/ Solve for t when $r(t)=(0,1,0)$

$2t=0$
$t=0$

Solve for t when $r(t)=\langle 2\pi,-1,0\rangle$

$2t=2\pi$
$t=\pi$

so $r_x'=2 \quad r_y'=-\sin(t), \quad r_z'=2\cos(t)$

$$L=\int_0^{\pi}\sqrt{4+\sin^2(t)+4\cos^2(t)}\,dt$$

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6) $r(t)=\langle 4\sin(t), 4\cos(t), 3t\rangle$

$r'(t)=\langle 4\cos(t), -4\sin(t), 3\rangle$

$r''(t)=\langle -4\sin(t), -4\cos(t), 0\rangle$

$a_T=\dfrac{r'\cdot r''}{|r'|} \qquad a_N=\dfrac{|r'\times r''|}{|r'|}$

$|r'|=\sqrt{16\cos^2(t)+16\sin^2(t)+9}=5$

$r'\cdot r''=-16\cos(t)\sin(t)+16\cos(t)\sin(t)=0$

So $a_T=0$

Hence

$r''(T) = a_N \vec{N} + 0\vec{T}$

$|r''(T)| = a_N \cdot |\vec{N}| = a_N$

$|r''(T)| = \sqrt{16(\sin^2(T)+\cos^2(T))} = 4$

$a_N = 4$

7/

$r'(t) = \langle 2t, 3t^2, 2t+1 \rangle \qquad r'(0) = \langle 0,0,1 \rangle$

$r''(t) = \langle 2, 6t, 2 \rangle \qquad r''(0) = \langle 2,0,2 \rangle$

Solve for t

$r(t) = \langle -1,0,0 \rangle$ so $t^3=0$ so $t=0$

~~$t^2-1=0$ so $t=\pm 1$~~

~~Osculating plane contains perp to r', r''~~

~~By examination r', r'' in xy p~~

Osculating plane ~~parallel~~ contains r', r''

so osculating plane paralel to ~~xy~~ xz plane,

So eq has form $y - d = 0$

$\langle 0,0,1 \rangle$ on plane $\quad 0-d=0$, $d=0$

So $\boxed{y=0}$ eq for plane

~~8~~ $r(t) = (1, 0, -1)$

$t = 1$

$r'(t) = \langle 1, 1, 2t \rangle \quad r'(1) = \langle 1, 1, 1 \rangle$

$\cancel{r''(t) = \langle 0, 0, 2 \rangle \quad r''(1) = \langle 0, 0, 2 \rangle}$

$|r'(t)| = \sqrt{2+4t^2}$

$$T(t) = \left\langle \frac{1}{\sqrt{2+4t^2}}, \frac{1}{\sqrt{2+4t^2}}, \frac{2t}{\sqrt{2+4t^2}} \right\rangle$$

$$T'(t) = \left\langle \frac{-4t}{(2+4t^2)^{\frac{3}{2}}}, \frac{-4t}{(2+4t^2)^{\frac{3}{2}}}, \cancel{\frac{\sqrt{2+4t^2}\cdot 2 - 8t^2\sqrt{2+4t^2}}{2+4t^2}} \right\rangle$$

~~$T'(t)$ paralel to $\langle -4t, -4t, (2-8t^2)(2+4t^2) \rangle$~~

~~$N(t)$ paralel to $T'(t)$ so~~

~~$N(1) = \langle -4, -4,$~~

$$T'(t) = \left\langle \frac{-4t}{(2+4t^2)^{\frac{3}{2}}}, \frac{-4t}{(2+4t^2)^{\frac{3}{2}}}, \frac{2}{\sqrt{2+4t^2}} + \frac{-2t \cdot 1 \cdot 4t}{(2+4t^2)\sqrt{2+4t^2}} \right\rangle$$

$$= \frac{\langle -4t, -4t, 2(2+4t^2) - 8t^2 \rangle}{\sqrt{2+4t^2}}$$

$T'(1)$ paralel to $\langle -4,-4,4\rangle$

$$|\langle -4,-4,4\rangle| = 4\cdot\sqrt{3}$$

$$N(t) = \frac{T'(1)}{T(1)} = \frac{\langle -1,-1,1\rangle}{\sqrt{3}}$$

9/ We have normals

$\langle -5, 1, 1\rangle$

$$\boxed{\langle 1,-1,0\rangle + \langle -5,1,1\rangle t}$$

$$\begin{aligned} x+2y&=-1\\ 2x-2y&=4\\ \hline 3x&=3,\quad x=1 \end{aligned}$$

$y=-1 \quad z=0$

10/ Volume of paralel piped =
abs value of scalar triple prod

$$\left|\langle 3,1,1\rangle \cdot \left(\langle 1,2,4\rangle \times \langle -1,5,5\rangle\right)\right|$$

$$\langle 1,2,4\rangle \times \langle -1,5,5\rangle = \langle -10,-9,7\rangle$$

$$\langle 3,1,1\rangle \cdot \langle -10,-9,7\rangle = -32$$

$$\boxed{\text{Volume} = 32}$$

11/ Find radius of sphere

$$x^2+6x+y^2-4y+z^2-2z=2$$

Must be of form $\rightarrow (x+3)^2+(y-2)^2+(z-1)^2=r^2$

*since to get linear terms to match

Expanding we get

~~$(x^2+6x$~~

$$(x^2+6x+y^2-4y+z^2-2z)+9+4+1=r^2$$

$$(x^2+6x+y^2-4y+z^2-2z)=r^2-14=2$$

$$r^2=16$$

$$\boxed{r=4}$$

L3 #12/

$$c = \frac{1}{\sqrt{2x - x^2 - y^2}}$$

$$c^2 = \frac{1}{2x - x^2 - y^2}$$

$$\frac{1}{c^2} = 2x - x^2 - y^2$$

$$x^2 - 2x + y^2 = -\frac{1}{c^2}$$

$$(x-1)^2 \cancel{-1} + y^2 = -\frac{1}{c^2}$$

$$(x-1)^2 + y^2 = 1 - \frac{1}{c^2}$$

$$r^2 = 1 - \frac{1}{c^2}$$

but $\frac{1}{c^2} \geq 0$ so $r^2 \leq 1$

so $r \leq 1$

Circles w/ $r \leq 1$ centered at $(1,0)$

13/ $\vec{a} \times \vec{b}$ perp to $\vec{a}, \vec{b}$ so

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ 4 & -1 & 3 \end{vmatrix} = \langle 7, 7, -7 \rangle = \langle 7, 7, -7 \rangle$$

$$|\langle 7, 7, -7 \rangle| = 7\sqrt{3}$$

$$\pm \frac{\langle 1, 1, -1 \rangle}{\sqrt{3}} = \boxed{\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle}$$

14/ b) Want point where

$r(t) = \langle 2+t, t, 3-3t \rangle$ isect

$3x - y + 2z = 4$

So

$3(2+t) - t + 2(3-3t) = 4$

$6 + 3t - t + 6 - 6t = 4$

~~$10 - 3t = 4$~~ ~~$8 = 3t$~~ $12 - 4t = 4$

$t = 2$

$\langle 2+2, 2, 3-6 \rangle = \boxed{\langle 4, 2, -3 \rangle \text{ Point of isect}}$

15/ a) M has normal vector $\langle 1, -3, 3 \rangle$

Normal plane has normal T

so also normal as $r'(t) = T \cdot |r'(t)|$

So when M and normal are paralel

$r'(t) = \langle 1, 2t, 3t^2 \rangle = c \langle 1, -3, 3 \rangle$

$c = 1$

$2t = -3$ but $3 \cdot \left(\frac{-3}{2}\right)^2 \neq 3$

$t = -\frac{3}{2}$

So no points

b/ For tangent line to be paralel to plane it must be $\perp$ to plane's normal.

So

$r'(t) \cdot \langle 1, -3, 3 \rangle = 0$

$1 - 6t + 9t^2 = 0$

$(1-3t)^2 = 0$

$\boxed{t = \frac{1}{3}}$ so

$r(\frac{1}{3}) = \boxed{\langle \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \rangle}$