Limits

Lets consider the following:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

If we approach along the line $x = 0$ or $y = 0$ the value of f goes to 0. Now lets try again along the line $y = mx$. Then

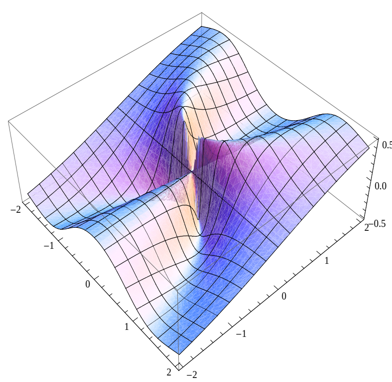
$$\frac{x^2y}{x^4+y^2} = \frac{mx^3}{x^4+m^2x^2} = \frac{mx}{x^2+m^2}.$$

So for $m \neq 0$ the value of f goes to 0. So far all approaches give the same value of 0. Lets try $y = x^2$. Then

$$\frac{x^2y}{x^4+y^2} = \frac{x^4}{x^4+x^4} = \frac{1}{2}.$$

So in this approach the value of f goes to $\frac{1}{2}$ not 0 and hence this limit does not exist.

It might be useful to look at the graph of $f(x,y)$:



Let try a similar limit.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^2}.$$

Now the limit exists. Why? Since y^2 and x^4 are always positive, $0 \leq x^4 \leq x^4 + y^2$. So $0 \leq \frac{x^4}{x^4+y^2} \leq 1$ and therefore $0 \leq \frac{x^4}{x^4+y^2} \leq y$. So as y goes to 0 so does $f(x,y)$. Hence $\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^2} = 0$. For a real challenge try $f(x,y) = \frac{x^4y}{x^5+y^2}$.