~~Teaching Journal~~ Scratch solutions for practice exam 1

1) Find an equation for the line thro the point $(3,-1,2)$ and perpindicular to the plane

$$2x-y+z+10=0$$

The plane has normal vec $\langle 2,-1,1\rangle$ $\perp$ to plane so

$r=\langle 3,-1,2\rangle+\langle 2,-1,1\rangle s$ is parametric eq

To get eq in just x,y,z we set

$$\frac{x-3}{2}=\frac{y+1}{-1}=\frac{z-2}{1}$$

a

Find the distance between the pt $(-1,-1,-1)$ and the plane $x+2y+2z-1=0$

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}, \text{ w/ } \langle a,b,c\rangle = \text{normal}$$

$$= \langle 1,2,2\rangle$$

$$\langle x_1,y_1,z_1\rangle = \langle -1,-1,-1\rangle$$

$$D = \frac{6}{\sqrt{1^2+4+4}} = \frac{6}{3} = \boxed{2}$$

Solution Notes For Prac Exam 1

4) Find the values of b s.t. the vetors $\langle -11, b, 2\rangle$ and $\langle b, b^2, b\rangle$ are orthogonal.

$\vec{v}\cdot\vec{w} = 0 \Leftrightarrow \vec{V} \perp \vec{w}$ so

$\langle -11, b, 2\rangle \cdot \langle b, b^2, b\rangle = -11b + b^3 + 2b = b^3 - 9b = 0$

$\Leftrightarrow$ perp $\Leftrightarrow$ $\cancel{b^3 = 9b}$

$b(b^2-9) = 0 \Leftrightarrow b = 0$ or $b^2 - 9 = 0$

$\Leftrightarrow \boxed{b = 0, 3, -3}$

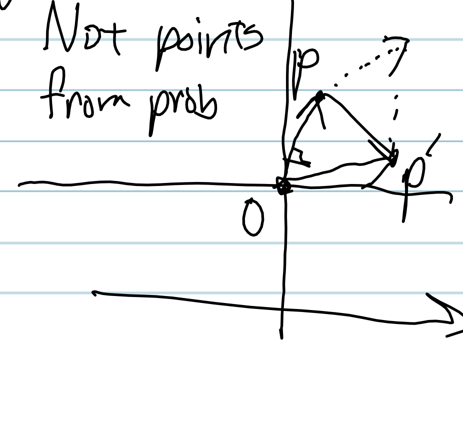
5/ Find the area of Δ w/ vertices at $(0,0,0)$, $(-1,0,-1)$ and $(1,-1,2)$

Let $O = (0,0,0)$, $P = (-1,0,-1)$, $\hat{P} = (1,-1,2)$

Want area of $\Delta\ OP\hat{P} = \frac{1}{2}$ area parallelogram det by $\vec{OP}$, $\vec{O\hat{P}}$

Area of $\vec{OP}\ \vec{O\hat{P}}$ paralelogram $= |\vec{OP} \times \vec{O\hat{P}}|$

Not points from prob



$\vec{OP}$

Area $\Delta\ OP\hat{P} = \frac{1}{2} |\langle -1, 0, -1\rangle \times \langle 1, -1, 2\rangle|$

$$= \frac{1}{2}\left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{vmatrix} \right| = \frac{1}{2}\left| 1\vec{i} - \vec{j}(+2 - -1) + \vec{k}\cdot 1 \right|$$

$$= \cancel{\frac{1}{2}|\langle 1, 1, 1\rangle|}$$

$$= \frac{1}{2}|\langle 1, -3, 1\rangle| = \frac{\sqrt{1+9+1}}{2} = \boxed{\frac{\sqrt{11}}{2}}$$

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6) Which vector is always orthogonal to $\vec{b} - \text{proj}_a \vec{b}$.

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}}$$

ortho $\Leftrightarrow$ dot prod $= 0$

$$\vec{a}\cdot(\vec{b} - \text{proj}_a \vec{b}) = \vec{a}\cdot b - \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}\cdot\vec{a} = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} = 0$$

$$\vec{b}\cdot(\vec{b} - \text{proj}_a \vec{b}) = \vec{b}\cdot\vec{b} - \vec{b}\cdot\text{proj}_{\vec{a}}\vec{b} = \vec{b}\cdot\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}\cdot\vec{b}$$

$$(\vec{a} - \vec{b})\cdot(\vec{b} - \text{proj}_{\vec{a}}\vec{b}) = $$

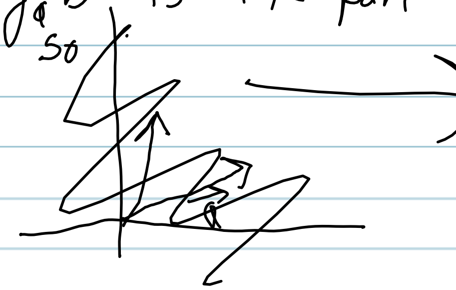
Something complicated.

But we already saw

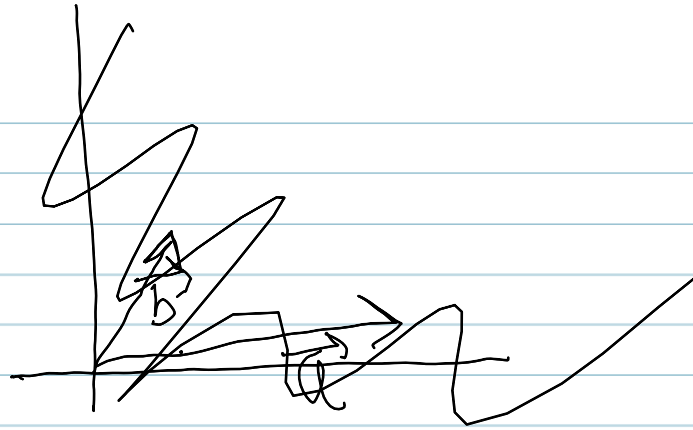
$\vec{a}\cdot(\vec{b} - \text{proj}_a \vec{b}) = 0$ so

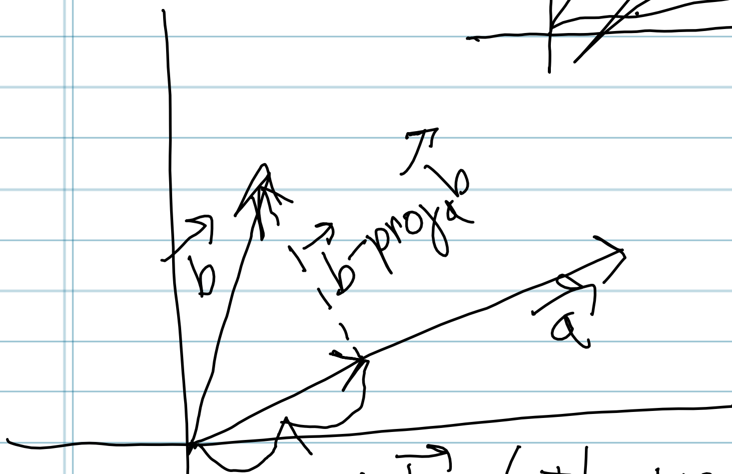
$$\boxed{\vec{b} - \text{proj}_a \vec{b} \text{ always orthogonal to } \vec{a}}$$

Alternativly observe visually that $\text{proj}_a \vec{b}$ is the part of $\vec{b}$ not $\perp$ to $\vec{a}$ so $\vec{b} - \text{proj}_a \vec{b}$ is just part ortho $\vec{a}$



6 continued





$\text{proj}_{\vec{a}}\vec{b}$ (the vector indicated NOT the length of that vector)

7/ Find the parametric eqs of the intersection of the planes $x-z=0$, $y=0$ and $x-y+2z+3=0$

The normal vector for the first plane is $\langle 1,0,-1\rangle$, to the second $\langle 1,-1,2\rangle$.
The dir $\vec{v}$ of our line $r = \vec{r_0} + \vec{v}s$ is $\perp$ to both so

let $\vec{V} = \langle 1,0,-1\rangle \times \langle 1,-1,2\rangle$

$$\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \boxed{-1\vec{i} - 3\vec{j} + -1\vec{k} = \langle -1,-3,-1\rangle}$$

Pick pt on both planes, $y=0, x=z \to 3z+3=1, z=-1$
$\langle 1,0,-1\rangle$

7/ cont

$r(s) = \langle -1, 0, -1 \rangle + \langle -1, -3, -1 \rangle s$

Not any of ans so see if we can convert it

Only options w/ paralell direction vec = a, b

Try a) $\hat{r}(t) = \langle 0, 3, 0 \rangle + \langle -1, -3, -1 \rangle t$

$\langle 0, 3, 0 \rangle = \langle -1, 0, -1 \rangle - \langle -1, -3, -1 \rangle$

$\hat{r}(t) = r(t-1)$ so $\boxed{a)}$

8/ Find eq for normal plane to

$\langle e^{t-1}, t^2, \cos(1-t) \rangle$, @ $t = 1$ | $r(0) = \langle 1, 1, 1 \rangle$

$r'(t) = \langle e^{t-1}, 2t, \sin(1-t) \rangle$

$r'(1) = \langle 1, 2, 0 \rangle$ same dir as T

Normal plane perp to T so perp to $r'(t)$

so has equation

$1x + 2y + d = 0$ but $r(0)$ on plane

~~$0x + 2 \cdot 0 = 0, z = 1$~~

$\boxed{\cancel{x + 2y = 0, z = 1}}$

$x + 2y + d = 0$
$1 + 2 + d = 0$
$d = -3$

$$\boxed{x+2y-3=0}$$

9) Eq of sphere center $(4,-1,3)$ radius $\sqrt{5}$

distance of (x,y,z) to $(4,-1,3)$ is

$$d(x,y,z)=\sqrt{(x-4)^2+(y+1)^2+(z-3)^2}$$

So set of points $\sqrt{5}$ away from $(4,-1,3)$ is

$$\boxed{5=(x-4)^2+(y+1)^2+(z-3)^2}$$

10) $r(t)=\langle t^2,-2t,\ln t\rangle$ gives pos $t>0$

Find a_T, a_N at $t=1$

$$a_N=\frac{|r'(t)\times r''(t)|}{|r'(t)|},\quad a_T=\frac{r'(t)\cdot r''(t)}{|r'(t)|}$$

$$r'(t)=\langle 2t,-2,\tfrac{1}{t}\rangle \qquad r'(1)=\langle 2,-2,1\rangle$$

$$r''(t)=\langle 2,0,-\tfrac{1}{t^2}\rangle \qquad r''(1)=\langle 2,0,-1\rangle$$

$$a_N(1)=\left(\langle 2,-2,1\rangle\times\langle 2,0,-1\rangle\right)/|\langle 2,-2,1\rangle|$$

$$\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-2&1\\2&0&-1\end{vmatrix}=\langle -2,4,4\rangle$$

So

$$a_N = \frac{|\langle -2,4,4\rangle|}{|\langle 2,-2,1\rangle|} = \sqrt{\frac{4+16+16}{4+4+1}} = \sqrt{\frac{36}{9}}$$

$$= \frac{6}{3} = 2 \qquad \boxed{a_N = 2}$$

$$a_T(1) = \frac{r'(1)\cdot r''(1)}{|r'(1)|} = \frac{\langle 2,-2,1\rangle \cdot \langle 2,0,-1\rangle}{|\langle 2,-2,1\rangle|}$$

$$= \frac{4+0+-1}{3} = \frac{3}{3} = 1 \qquad \boxed{a_T = 1}$$

11/ Volume of paralell piped given by $\langle 1,2,7\rangle, \langle 0,-3,4\rangle, \langle 0,0,6\rangle$

Volume given by (absolute value of) scalar triple prod

$$|\langle 1,2,7\rangle \cdot (\langle 0,-3,4\rangle \times \langle 0,0,6\rangle)|$$

Use Trick to compute faster

$$= |\langle 1,2,7\rangle \cdot ([\langle 0,-3,0\rangle + \langle 0,0,4\rangle] \times \langle 0,0,6\rangle)|$$

$$= |\langle 1,2,7\rangle \cdot (\langle 0,-3,0\rangle \times \langle 0,0,6\rangle + \langle 0,0,4\rangle \times \langle 0,0,6\rangle)|$$

$$= \cancel{\langle 1,2,7\rangle} \cdot$$

$= \left| \langle 1,2,7 \rangle \cdot \left(\langle -18,0,0 \rangle + \langle 0,0,0 \rangle \right) \right|$

$= |-18| = \boxed{18}$

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12) $f(x,y) = e^{\sin(x)} + x^5 y + \ln(1+y^2)$

$$\frac{df}{dx} = e^{\sin(x)} \cos(x) + 5x^4 y + 0$$

$$\frac{d\left[\frac{df}{dx}\right]}{dy} = 0 + 5x^4 + 0$$

$$\boxed{\frac{d^2 f}{dx\,dy} = 5x^4}$$

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13) Find length of curve $r(t) = \langle -\ln(t), t^2, 2t \rangle$

$1 \leq t \leq e$

$$L = \int_1^e \sqrt{(r_x')^2 + (r_y')^2 + (r_z')^2}\,dt$$

$r_x' = -\frac{1}{t}$ $\quad r_z' = 2$

$r_y' = 2t$

$$= \int_1^e \sqrt{\frac{1}{t^2} + \cancel{2}4 + 4t^2}\,dt = \int_1^e \sqrt{\left(\frac{1}{t} + 2t\right)^2}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right)dt$$

$$= \ln|t| + t^2 \Big|_1^e = \ln|e| + e^2 - (\ln|1| + 1^2) = \boxed{e^2}$$

14/ Find the unit normal vector N(t) as a func of t for

$r(t) = \langle 3t, 4\cos(t), 4\sin(t) \rangle$

$r'(t) = \langle 3, -4\sin(t), 4\cos(t) \rangle$

$|r'(t)| = \sqrt{9 + 16\sin^2(t) + 16\cos^2(t)} = \sqrt{9+16} = \sqrt{25} = 5$

$T(t) = \frac{r'(t)}{|r'(t)|} = \cancel{\langle \frac{3}{5}, \frac{-4\sin(t)}{5}, 4\cos(t) \rangle}$

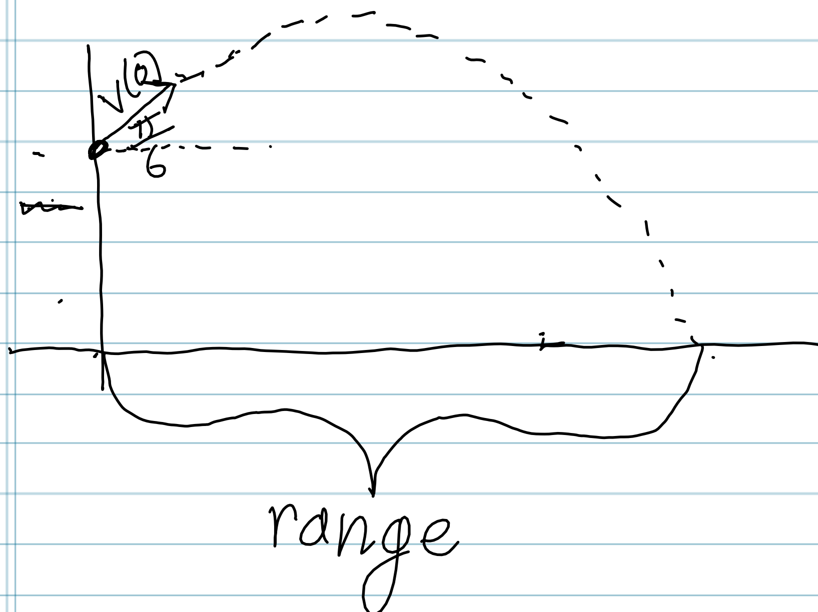
$\frac{\langle 3, -4\sin(t), 4\cos(t) \rangle}{5}$

$T'(t) = \frac{\langle 0, -4\cos(t), 4\sin(t) \rangle}{5}$ (5 is constant so don't need quotient rule)

$|T'(t)| = \sqrt{\frac{16(\sin^2(t) + \cos^2(t))}{5^2}} = \frac{4}{5}$

$N(t) = \frac{\frac{\langle 0, -4\cos(t), 4\sin(t) \rangle}{5}}{\frac{4}{5}} = \boxed{\langle 0, -\cos(t), \sin(t) \rangle}$

15/ Proj fired w/ angle $\frac{\pi}{6}$ at $2\,m/6$ from 4m above ground.
Find range if $g = 10\,m/s^2$



$r(t) =$ position $\qquad v(t) =$ velocity

$r(0) = \langle 0, 4 \rangle \qquad v(0) = \langle 2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6} \rangle$

$a(t) = \langle 0, -10 \rangle \qquad = \langle \sqrt{3}, 1 \rangle$

$v(t) = \int a(t)\,dt = \int \langle 0, -10 \rangle dt = \langle 0, -10t \rangle + \vec{c}$

$V(0) = \langle 0, -10\cdot 0 \rangle + \vec{c} = \langle \sqrt{3}, 1 \rangle$

$\langle 0, 0 \rangle + \vec{c} = \langle \sqrt{3}, 1 \rangle$

$\vec{c} = \langle \sqrt{3}, 1 \rangle$

$v(t) = \langle \sqrt{3}, 1 - 10t \rangle$

$$r(t) = \int v(t)\,dt = \int \langle \sqrt{3}, 1 - 10t \rangle\,dt$$

$$\cancel{= \langle \sqrt{3}t, -5t^2 \rangle + \vec{c}} = \langle \sqrt{3}t, t - 5t^2 \rangle + \vec{c}$$

$$\cancel{r(0) = \langle 0, 4 \rangle = \langle \sqrt{3}\cdot 0, -5\cdot 0^2 \rangle + \vec{c}}$$

$$\cancel{\langle 0, 4 \rangle = \vec{c}}$$

$$\cancel{r(t) = \langle \sqrt{3}t, 4 - 5t^2 \rangle}$$

~~range is x position when y position = 0~~

$$\cancel{0 = 4 - 5t^2}$$

$$r(0) = \langle \sqrt{3}\cdot 0, 0 - 5\cdot 0^2 \rangle + \vec{c} = \langle 0, 4 \rangle$$

$$\vec{c} = \langle 0, 4 \rangle$$

$$r(t) = \langle \sqrt{3}t, -5t^2 + t + 4 \rangle$$

range is x-pos when y-pos = 0

$-5t^2 + t + 4 = 0$ root at $t = 1$

so $\boxed{\text{range} = \sqrt{3}\cdot 1 = \sqrt{3}}$