Practice Exam 2

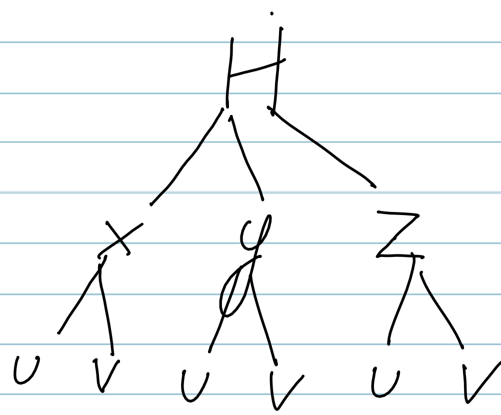
1/ $f(x,y) = \int_y^x e^{-t^2}\,dt$

$f(x,y) = F(x) - F(y)$ where $F(x) = \int_0^x e^{-t^2}\,dt$

$f_y = -e^{-y^2}$ Fund thm

$f_{yy} = 2y\,e^{-y^2}$

2/ $H = xe^{y-z^2}$



$\cancel{\frac{dH}{du} = \frac{dH}{dx}\cdot\frac{dx}{dx} + \frac{dH}{dy}\,d}$

$$\frac{dH}{du} = \frac{dH}{dx}\cdot\frac{dx}{du} + \frac{dH}{dy}\cdot\frac{dy}{du} + \frac{dH}{dz}\cdot\frac{dz}{du}$$

$\frac{dH}{dx} = e^{y-z^2}$ $\quad \frac{dH}{dy} = xe^{y-z^2}$ $\quad \frac{dH}{dz} = -2zx\,e^{y-z^2}$

$\frac{dx}{du} = 2v$ $\quad \frac{dy}{du} = 1$ $\quad \frac{dz}{du} = 1$

when $u=3$, $v=-1$

$x=-6 \quad y=4 \quad z=2 \quad y-z^2=0$

$\frac{dx}{du}=-2 \quad \frac{dH}{dx}=1 \quad \frac{dH}{dy}=-6 \quad \frac{dH}{dz}=24$

$\frac{dH}{du}=1\cdot-2+-6\cdot1+24\cdot1=16$

3/ $\frac{\vec{v}}{|\vec{v}|}=\frac{a\langle 1,2,-2\rangle}{\sqrt{1+4+4}}=\langle \frac{1}{3},\frac{2}{3},-\frac{2}{3}\rangle$

$\nabla f=\langle \sin(y,z), xz\cos(yz), xy\cos(yz)\rangle$

$\nabla f|_{\langle 1,3,0\rangle}=\langle \sin(0), 0\cdot\cos(0), 3\cdot\cos(0)\rangle$

$=\langle 0,0,3\rangle$

$\nabla f|_{\langle 1,3,0\rangle}\cdot\langle \frac{1}{3},\frac{2}{3},-\frac{2}{3}\rangle = 0+0+\frac{-2}{3}\cdot 3=\boxed{-2}$

A $\boxed{4}$ $f(x,y)=\sin(xy)$

$\nabla f=\langle y\cos(xy), x\cos(xy)\rangle$

$\nabla f|_{\langle 1,0\rangle}=\langle 0,1\rangle$ ~~max~~ dir of max change

$\nabla f\cdot\frac{\nabla f}{|\nabla f|}=1$

34

x, y, z

$g(x,y,z) = 4x + 4y + 4z = 24$

$f(x,y,z) = xyz$

Lagrange multipliers

$\nabla f = \langle yz, xz, yx \rangle$

$\nabla g = \langle 4, 4, 4 \rangle$

$\nabla f = \lambda \nabla g$

$yz = \lambda 4$

$xz = \lambda 4 \rightarrow$ $yz = xz$, $xz = xy$

$xy = \lambda 4$

~~Either~~ If $x, y,$ or $z = 0$ $\lambda = 0$ so $x = y = z = 0$ and $g(x,y,z) \neq 24$

So $y = x$, $z = y$ $\quad x = y = z$

$12x = 24$

$x = 2$

$y = 2$

$z = 2$

$$\boxed{f(x,y,z) = 8}$$

$\cancel{A5}$

$$D = f_{xx} f_{yy} - f_{xy}^2$$
$$= 3 \cdot 2 - 2^2$$
$$= 6 - 4 = 2 > 0$$
$$f_{xx} = 3 > 0$$

local min

$\cancel{B5}$

$$\nabla f = \langle yz, xz, yx \rangle$$
$$g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6$$
$$\nabla g = \langle 2x, 4y, 6z \rangle$$
$$\nabla f = \lambda \nabla g$$
$$yz = \lambda 2x$$
$$xz = \lambda 4y$$
$$yx = \lambda 6z$$

If x, y or $z = 0$ then either $x = y = z = 0$ and $g(x,y,z) \neq$ or exactly 2 of x, y, z are 0

Either $x = 0$ or $\lambda = \frac{yz}{2x}$

$y = 0$ or $\lambda = \frac{xz}{4y}$

$z = 0$ or $\lambda = \frac{yx}{6z}$

$$\frac{yz}{2x} = \frac{xz}{4y}$$

$$4y^2z = 2x^2z$$

$$2y^2 = x^2$$

$$\frac{yx}{6z} = \frac{yz}{2x}$$

$$y2x^2 = \cancel{6}y6z^2$$

$$x^2 = 3\cancel{y}z^2$$

$$x^2 + x^2 + x^2 = 6$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$2y^2 = x^2 \quad y = \pm 1$$

$$z = \pm\sqrt{\frac{2}{3}}$$

PI	$f(x,y,z)$
$(0,0,\pm\sqrt{2})$	0
$(0,\pm\sqrt{3},0)$	0
$(\pm\sqrt{6},0,0)$	0
$(\pm\sqrt{2},\pm 1,\pm\sqrt{\frac{2}{3}})$	$\pm\frac{2}{\sqrt{3}}$

~~Mam~~ $\boxed{\text{Max value } \frac{2}{\sqrt{3}}}$

A6 Normal to tangent plane for level curve given by gradient

~~the~~ $g(x,y,z) = \frac{x^2}{9} + y^2 + \frac{z^2}{4}$

$$\nabla g\Big|_{\langle 3,-1,2\rangle} = \left\langle \frac{2}{9}x, 2y, \frac{z}{2} \right\rangle$$

$$= \left\langle \frac{2}{3}, -2, 1 \right\rangle$$

$$\frac{2}{3}(x-3) + -2(x+1) + 1(z-2) = 0$$

$$\boxed{\frac{2}{3}x - 2y + z - 6 = 0}$$

$\boxed{\text{B6}}$ Find the volume of the solid bounded by $z = 6 - xy$, $x = 2$, $x = -2$, $y = 0$, $y = 3$, $z = 0$

Let $R = \{(x,y) \mid -2 \le x \le 2,\ 0 \le y \le 3\}$

then

$$V = \iint_R (6 - xy)\, dA = \int_{-2}^{2}\int_{0}^{3} 6 - xy\, dy\, dx$$

$$= \int_{-2}^{2} \left(6y - \frac{xy^2}{2}\right)\Big|_0^3 dx = \int_{-2}^{2} 18 - \frac{9}{2}x\, dx = 72$$

B7

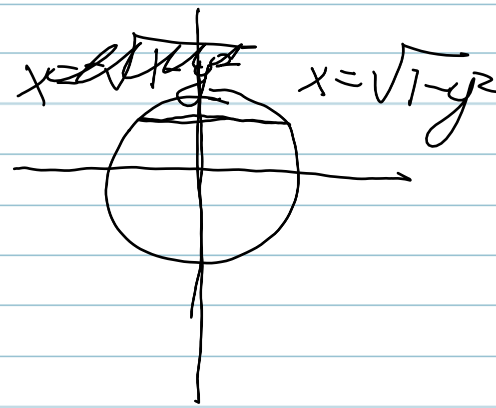
$$\iint_D \frac{2y}{x^2+1}\,dA$$

$$\int_0^1\int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \frac{y^2}{x^2+1}\Big|_0^{\sqrt{x}}\,dx$$

$$= \int_0^1 \frac{x}{x^2+1}\,dx = \frac{\ln|x^2+1|}{2}\Big|_0^1 = \frac{1}{2}\ln 2$$

A8

$$\iint_D \sqrt{1-y^2}\,dx\,dy$$

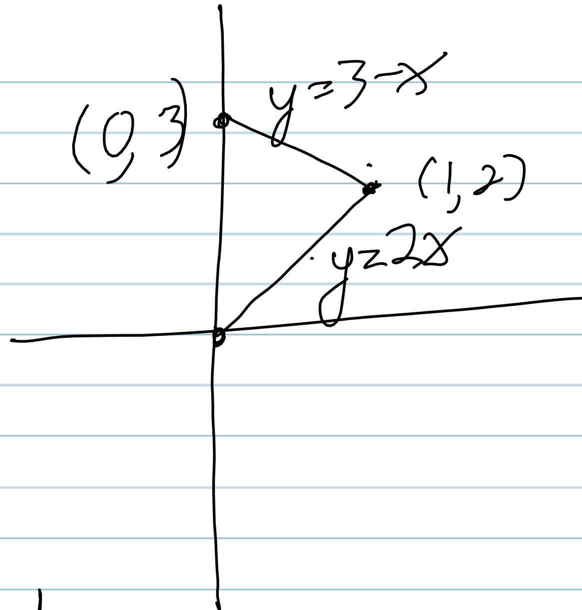


$$\int_0^1\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\,dy$$

$$\int_0^1 x\sqrt{1-y^2}\Big|_0^{\sqrt{1-y^2}}\,dy = \int_0^1 (1-y^2)\,dy$$

$$= y-\frac{1}{3}y^3\Big|_0^1 = \frac{2}{3}$$

B8



$$\int_0^1\int_{2x}^{3-x} 2xy\,dy\,dx$$

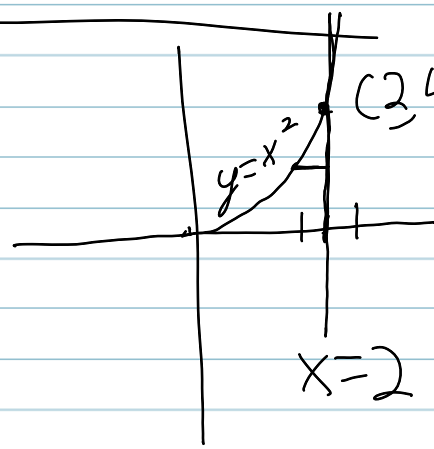
$$=\int_0^1 xy^2\Big|_{2x}^{3-x}\,dx = \int_0^1 x(3-x)^2 - x(2x)^2\,dx$$

$$=\int_0^1 (9x-6x^2+x^3-4x^3)\,dx = \int_0^1 (-3x^3-6x^2+9x)\,dx$$

$$=\frac{7}{4}$$

9

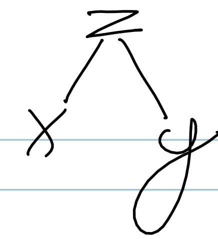
$$\int_0^4\int_{\sqrt{y}}^{2} \sqrt{x^3+1}\,dx\,dy$$



$$\int_0^2\int_0^{x^2} \sqrt{x^3+1}\,dy\,dx$$

$$=\int_0^2 x^2\sqrt{x^3+1}\,dx = \frac{2(x^3+1)^{3/2}}{9}\Big|_0^2 = \frac{52}{9}$$

10 Diff by x



$$4x^3 + 0 + 4z^3 \cdot \frac{dz}{dx} + 4yz + 4xy\frac{dz}{dx} = 0$$

$$x^3 + yz = -(z^3 + xy)\frac{dz}{dx}$$

$$\frac{dz}{dx} = -\frac{x^3 + yz}{z^3 + xy}$$

11/ i) Paralel when normal vectors $\parallel$

$$f(x,y,z) = x^2 + y^2 + z^2 = 9$$

$$\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \langle 2, -1, 2 \rangle$$

$$2x = \lambda 2$$

$$2y = \lambda - 1$$

$$2z = \lambda 2$$

$$-2y = \lambda$$

$$x = \lambda$$

$$2x = 2z$$

$$x = z$$

$$y = -\frac{x}{2}$$

$$x^2 + \left(\frac{-x}{2}\right)^2 + x^2 = 9$$

$$2.5x^2 = 9 \quad x^2 = \frac{18}{5} \quad x = \pm\sqrt{\frac{18}{5}} \approx \pm 3\sqrt{}$$

$$\left(3\sqrt{\frac{2}{5}}, -\frac{3}{2}\sqrt{\frac{2}{5}}, 3\sqrt{\frac{2}{5}}\right)$$

$$\left(-3\sqrt{\frac{2}{5}}, \frac{3}{2}\sqrt{\frac{2}{5}}, -3\sqrt{\frac{2}{5}}\right)$$

ii) $r(t) = \left(3\sqrt{\frac{2}{5}}, -\frac{3}{2}\sqrt{\frac{2}{5}}, 3\sqrt{\frac{2}{5}}\right) + \langle 2, -1, 2\rangle t$

(normal line goes thru both points)

12) Find absolute max/min

$\nabla f = \langle 2, -1 \rangle$

No critical points

Look @ boundary

$g(x,y) = x^2 + \frac{y^2}{4} = 4$

$\nabla g = \langle 2x, \frac{y}{2} \rangle$

$\nabla f = \lambda \nabla g$

$2 = \lambda 2x$

$-1 = \lambda \frac{y}{2}$

$$\frac{1}{x} = \lambda$$

$$-\frac{2}{y} = \lambda$$

$$\frac{1}{x} = -\frac{2}{y}$$

$$y = -2x$$

$$x^2 + \frac{y^2}{4} = 2$$

$$x^2 + \frac{(-2x)^2}{4} = 2$$

$$2x^2 = 2$$

$$x = \pm 1$$

PT	$f(x,y)$
$(1,-2)$	4
$(-1,2)$	-4

$$\boxed{\begin{array}{l} \text{Max value} = 4 \quad @ \quad (1,-2) \\ \text{Min value} = -4 \quad @ \quad (-1,2) \end{array}}$$

$\boxed{13}$

$$f(x,y,z) = 2x\,2y\,2z$$
$$= 8xyz$$

$$\nabla f = \lambda \nabla g$$

$$8yz = \lambda 18x \qquad \text{Obviously } x,y,z \neq 0$$
$$8xz = \lambda 72y$$
$$8xy = \lambda 8z$$

$$\lambda = \frac{xy}{z} \qquad \lambda = \frac{xz}{9y} \qquad \lambda = \frac{4yz}{9x}$$

$$9x^2y = 4yz^2$$
$$9x^2 = 4z^2$$
$$9y^2 = z^2$$
$$36y^2 = 4z^2$$

$$4z^2 + 4z^2 + 4z^2 = 108$$
$$3z^2 = 54$$
$$z^2 = 18$$
$$z = \pm 3\sqrt{2}$$

$$y = \frac{\sqrt{2}}{3}$$
$$x = \frac{2\sqrt{2}}{3}$$

$$V = 8 \cdot 2 \cdot \frac{2}{3}\sqrt{2}$$
$$\boxed{= \frac{32}{3}\sqrt{2}}$$

$g(x,y) = xy = 1$
$h(x,y) = y^2 + z^2 = 1$

$\boxed{14}$ $\cancel{\nabla f}$

$\nabla f = \langle y, z+x, y \rangle$

$\nabla g = \langle y, x, 0 \rangle$

$\nabla h = \langle 0, 2y, 2z \rangle$

$\nabla f = \lambda \nabla g + u \nabla h$

$\cancel{\nabla h}$

$y = \lambda y$

$z + x = \lambda x + u2y$

$y = u2z$

$1 = \frac{y}{y} = \lambda$ or $y = 0$ but $\cancel{g(x)}$ $xy = 1$ so $y \neq 0$

$z + x = x + u2y$ $\to$ $z = u2y$

$y = u2z$ $\quad$ $y = u2z$

$\frac{z}{2y} = u$, $\frac{y}{2z} = u$ or $z = 0$ ~~BUT~~

$\frac{z}{2y} = \frac{y}{2z}$ $\quad$ $2z^2 = 2y^2$

Hence either

$z=0$ so

$y=\pm 1$

$x=\pm 1$

Or

$y^2=z^2$

$2z^2=1$

$z=\pm\frac{1}{\sqrt{2}}$

$y=\pm\frac{1}{\sqrt{2}}$

$x=\pm\frac{2}{\sqrt{2}}$

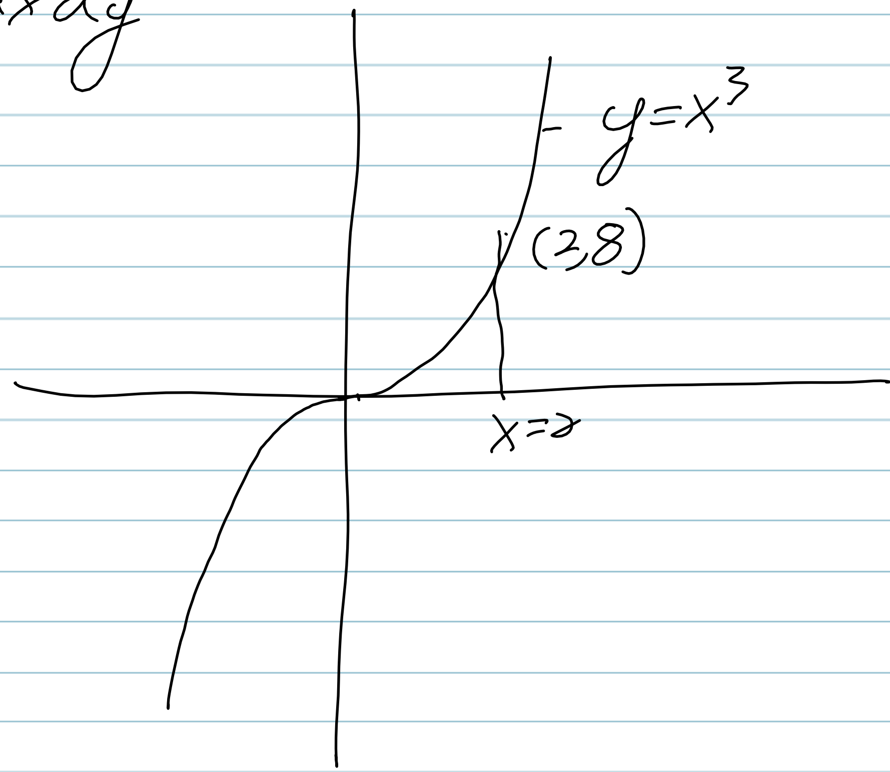
PI	$f(x,y,z)$
$(1,1,0)$	1
$(-1,-1,0)$	1
$\left(\frac{2}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	1.5
$\left(-\frac{2}{\sqrt{2}},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	.5
$\left(\frac{2}{\sqrt{2}},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$	.5
$\left(-\frac{2}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$	1.5

Max 1.5

Min .5

$\boxed{15}$ $\int_0^8 \int_{y^{\frac{1}{3}}}^2 e^{x^4}\,dx\,dy$

Switch order



$$\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 x^3 e^{x^4}\,dx$$

$u = x^4$

$du = 4x^3\,dx$

$\frac{du}{4} = x^3\,dx$

$$= \int e^u \frac{du}{4} = \frac{e^u}{4} = \frac{e^{x^4}}{4}\Big|_0^2 = \boxed{\frac{e^{16}}{4} - \frac{1}{4}}$$

A7

$$\iint_R 4+x^2-y^2\,dA$$

$$=\int_{-1}^{1}\int_{0}^{2}(4+x^2-y^2)\,dy\,dx$$

$$\cancel{=\int_{-1}^{1}\left(4+x^2-\tfrac{1}{3}y^3\right)\Big|_{y=0}^{y=2}}$$

$$=\int_{-1}^{1}\left(4y+x^2y-\tfrac{1}{3}y^3\right)\Big|_{y=0}^{y=2}$$

$$=\int_{-1}^{1} 8+2x^2-\frac{8}{3}\,dx$$

$$\cancel{=\int_{-1}^{1}\left(8-\tfrac{8}{3}\right)dx}^{\,0}+\cancel{\int_{-1}^{1}2x^2\,dx}$$

$$=8x+\frac{2}{3}x^3-\frac{8}{3}x\Big|_{-1}^{1}$$

$$=16+\frac{4}{3}-\frac{16}{3}=16-\frac{12}{3}=12$$