Practice Exam III Sols

1/ $r = 2\cos 3\theta$

$r = 0$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}$

$$\iint 1\, dA = \int_{\pi/6}^{\pi/2} \int_0^{2\cos 3\theta} 1\, r\, dr\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \frac{1}{2} r^2 \Big|_0^{2\cos 3\theta} = \int_{\pi/6}^{\pi/2} \frac{1}{2} 4\cos^2 3\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (2\cos^2 3\theta - 1 + 1)\, d\theta = \int_{\pi/6}^{\pi/2} (\cos 6\theta + 1)\, d\theta$$

$$= \frac{1}{6}\sin 6\theta + \theta \Big|_{\pi/6}^{\pi/2} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$$

$\boxed{2}$ $z = 4 - x^2 - y^2$

$$\int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} 2r^2 - \frac{1}{4}r^4 \Big|_0^2 d\theta = \int_0^{2\pi} 8 - \frac{16}{4}$$

$4 \cdot 2\pi = 8\pi$

$\boxed{3}$ $\quad p(x,y) = 2\sqrt{x^2+y^2} \qquad x^2+y^2=\pi^2,\ y \geq 0$

By symetry $\bar{x} = 0$

$$m = \int_0^{\pi}\int_0^{\sqrt{\pi}} 2r \cdot r\,dr\,d\theta = \int_0^{\pi} \frac{2}{3}\pi^{3}\,d\theta = \frac{2}{3}\pi^4$$

$$M_x = \int_0^{\pi}\int_0^{\sqrt{\pi}} r\sin\theta \cdot 2r^2\,dr\,d\theta = \int_0^{\pi} \frac{1}{2}r^4\sin\theta \Big|_0^{\sqrt{\pi}} d\theta$$

$$= \int_0^{\pi} \frac{\pi^4}{2}\sin\theta\,d\theta = -\cos(\theta)\frac{\pi^4}{2}\Big|_0^{\pi} = \pi^4$$

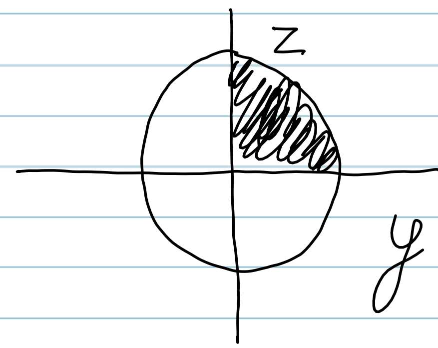
$$\bar{y} = \frac{M_x}{m} = \frac{\pi^4}{\frac{2}{3}\pi^4} = \frac{3}{2}$$

$$\boxed{\left(0, \frac{3}{2}\right)}$$

4/ $\iiint_E z\,dV$

where E bounded by $y^2+z^2=9$, $x=0$, $y=3x$ and $z=0$ first octant

$x = \frac{y}{3}$



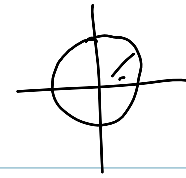
$$\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{y/3} z\,dx\,dy\,dz$$

$$= \int_0^3\int_0^{\sqrt{9-z^2}} z\frac{y}{3}\,dy\,dz$$

$$= \int_0^3 \frac{zy^2}{6}\Big|_0^{\sqrt{9-z^2}} = \int_0^3 \frac{z(9-z^2)}{6} = \int_0^3 \frac{9z-z^3}{6}\,dz$$

$$= \frac{9}{12}z^2 - \frac{z^4}{24}\Big|_0^3 = \frac{9\cdot9}{12} - \frac{9\cdot9}{24} = \frac{9\cdot9}{24} = \boxed{\frac{27}{8}}$$

$\boxed{5}$ $$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$$



$$\int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} r\cos\theta\, z\, r\,dz\,dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{2} \frac{z^2}{2} r^2\cos\theta \Big|_{r}^{2} dr\,d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{2} \left(2r^2\cos\theta - \frac{r^4}{2}\cos\theta\right) dr\,d\theta$$

$$= \int_{0}^{2}\int_{0}^{2\pi} \left(2r^2 - \frac{r^4}{2}\right)\cos\theta\,d\theta\,dr$$

$$= \int_{0}^{2} \left(2r^2 - \frac{r^4}{2}\right)\sin\theta \Big|_{0}^{2\pi} dr$$

$$= \int_{0}^{2} 0\,dr = \boxed{0}$$

$\boxed{6}$ $\iiint_E z\, dV$ where E lies between $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in first quadrant.

$z = \rho\cos\phi \qquad dA = \rho^2\sin\phi$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^{2} \rho\cos\phi\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{4}(2^4-1^4)\cos\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{15}{8}\sin^2\phi\Big|_0^{\frac{\pi}{2}} d\theta = \int_0^{\frac{\pi}{2}} \frac{15}{8}\, d\theta$$

$$= \boxed{\frac{15}{16}\pi}$$

7/

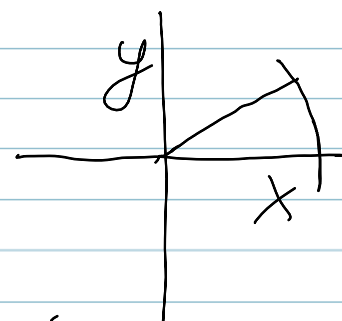
$$\cancel{\int_{-\pi/2}^{\pi/2}\int_0^3\int_0^{\pi/6} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta}$$

$$\cancel{\int_{-\pi/2}^{\pi/2}\int_0^3 \left(1-\frac{\sqrt{3}}{2}\right)\rho^2 \, d\rho \, d\theta}$$

$$\cancel{\int_0^{\pi} \left(1-\frac{\sqrt{3}}{2}\right)\frac{\rho^3}{3}\Big|_0^3 = \int_0^{\pi} 9\left(1-\frac{\sqrt{3}}{2}\right) d\theta}$$

$$\cancel{= \int_{-\pi/2}^{\pi/2}\int_0^3 -\rho^2\cos\phi \Big|_0^{\pi/6} =}$$

$$\int_0^{\pi/6}\int_0^3\int_0^{\pi} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$



$$= \int_0^{\pi/6}\int_0^3 2\rho^2 \, d\rho \, d\theta = \int_0^{\pi/6} 18 \, d\theta = \boxed{3\pi}$$

Be careful if you try ϕ goes between 0 and $\frac{\pi}{6}$ you get a cone not a wedge

$\boxed{8}$

$$\boxed{T:\ x=\frac{1}{4}u+\frac{1}{4}v,\quad y=\frac{1}{4}u-\frac{3}{4}v}$$

$$x=\frac{1}{4}u+\frac{1}{4}v$$

$$-\ y=\frac{1}{4}u-\frac{3}{4}v$$

$$x-y=v$$

$$v=x-y$$

$$\cancel{x=\frac{1}{4}x-\frac{1}{4}y+\frac{1}{4}u}$$

$$x=\frac{1}{4}u+\frac{1}{4}x-\frac{1}{4}y$$

$$4x-x+y=u$$

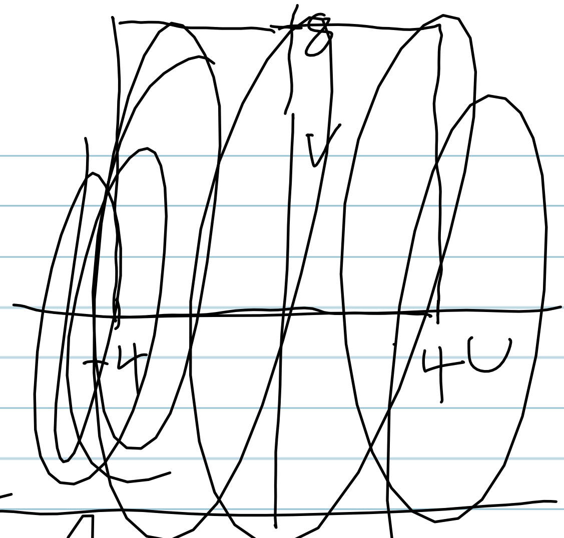
$$T^{-1}\ \boxed{u=3x+y,\quad v=x-y}$$

This maps straight lines to straight lines so just look at image of points

$T^{-1}(-1,3)=(0,-4)$ $\quad T^{-1}(1,5)=(8,-4)$

$T^{-1}(1,-3)=(0,4)$

$T^{-1}(3,-1)=(8,4)$



$$\iint_R x-2y\,dA$$

$$=\iint_{T^{-1}(R)}\left(\frac{1}{4}u+\frac{1}{4}v-\frac{1}{2}u+\frac{3}{2}v\right)\begin{vmatrix}\frac{dx}{du} & \frac{dx}{dv}\\ \frac{dy}{du} & \frac{dy}{dv}\end{vmatrix}$$

$$\begin{vmatrix}\frac{dx}{du} & \frac{dx}{dv}\\ \frac{dy}{du} & \frac{dy}{dv}\end{vmatrix}=\begin{vmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & -\frac{3}{4}\end{vmatrix}$$

$$=\int_{-4}^{4}\int_{0}^{8}\left(-\frac{1}{4}u+\frac{7}{8}v\right)-\frac{1}{4}\,dv\,du$$

$$=\int_{-4}^{4}\frac{1}{4}\left(-2u+\frac{7\cdot 8}{2}\right)du$$

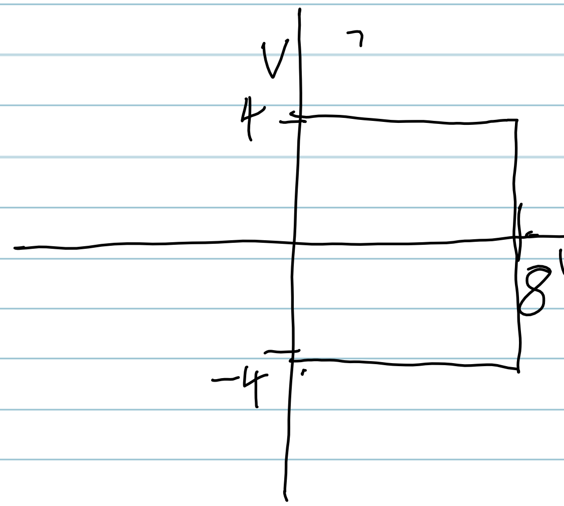
$$=\left.\left(-u^2+\frac{7\cdot 8}{2}u\right)\right|_{-4}^{4}$$

$$=\int_{-4}^{4}\left(\frac{u}{2}-7\right)du$$

$\frac{dx}{du}=\frac{1}{4}$ $\frac{dx}{dv}=\frac{1}{4}$ $\iint_R (x-2y)\,dA$

$\frac{dy}{du}=\frac{1}{4}$ $\frac{dy}{dv}=-\frac{3}{4}$ $=\int_{-4}^{4}\int_{0}^{8}\left[\left(\frac{1}{4}u+\frac{1}{4}v\right)-2\left(\frac{1}{4}u-\frac{3}{4}v\right)\right]\left|-\frac{1}{4}\right|du\,dv$

$\begin{vmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & -\frac{3}{4}\end{vmatrix}=\left|-\frac{1}{4}\right|=\boxed{-16}$



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9 $x=4u$ $y=2v$ $z=3w$

$\frac{x^2}{16}+\frac{y^2}{4}+\frac{z^2}{9}\le 1$

$u^2+v^2+w^2\le 1$

$\iiint_E z\,dV=\iiint_{\text{unit sphere}} 3w\begin{vmatrix}4&0&0\\0&2&0\\0&0&3\end{vmatrix}\,\cancel{du\,dv}\,dw\,du\,dv$

$=\int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} 72w\,dw\,du\,dv=0$ by symetry

$\boxed{10}$ $\int_C (x^2y^3-\sqrt{x})\,dy$

$r(t)=\langle t^2, t\rangle$

$0 \le t \le 2$

$$\int_0^2 \left[(t^2)^2 t^3 - t\right] \frac{dy}{dt}\,dt$$

$$= \int_0^2 (t^7-t)\cdot 1\,dt = \frac{1}{8}t^8 - \frac{1}{2}t^2 \Big|_0^2$$

$$= 32-2 = \boxed{30}$$

$\boxed{11}$ $\int_C xz\,ds$

$r(t)=\langle 4\sin t, 3t, 4\cos t\rangle$

$0 \le t \le \pi$

$$\int_0^\pi 4\sin(t)\cdot 4\cos(t)\cdot\sqrt{(4\cos(t))^2+3^2+(-4\sin(t))^2}\,dt$$

$$\int_0^\pi 80\sin(t)\cos(t)\,dt$$

$$= 40\sin^2(t)\Big|_0^\pi = 0$$

12

$$\int_C \tan(y)\,dx + x(\sec(y))^2\,dy$$

$$= \int_C F\cdot dr \qquad F = \langle \tan(y),\ x\sec^2(y)\rangle$$

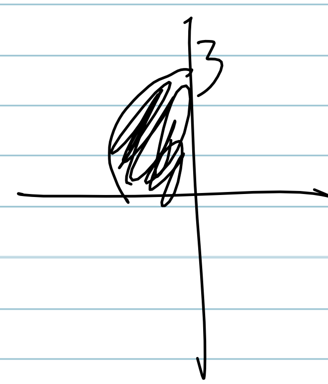
$$F = \nabla f, \quad f = x\tan(y)$$

$$\int_C F\cdot dr = f\left(2,\tfrac{\pi}{4}\right) - f(0,1)$$

$$= 2-0 = \boxed{2}$$

$\boxed{13}$

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{0} x^2 y\,dx\,dy$$



$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^3 r^2\cos^2\theta\, r\sin\theta\, r\,dr\,d\theta$$

$$= \frac{81}{5}$$

$\boxed{14}$

$x+2y+z=4$

$z=4-x-2y$

$$\int_0^2\int_0^{4-2y}\int_0^{4-x-2y} 1\,dz\,dx\,dy$$

$x+2y=4$

$x=4-2y$

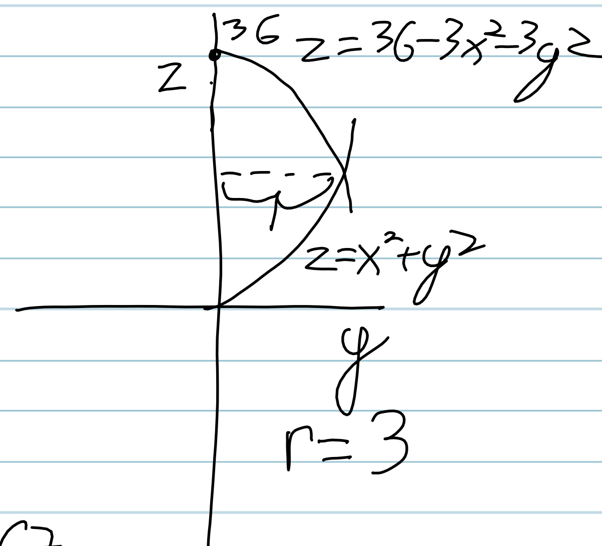
$$=\frac{16}{3}$$

$\boxed{15}$

$z=x^2+y^2$

$z=36-3x^2-3y^2$

$\bar{x}=\bar{y}=0$ by symetry



$$m=\int_0^{2\pi}\int_0^{3}\int_{r^2}^{36-3r^2} 1r\,dz\,dr\,d\theta=243\pi$$

$$\bar{z}=\frac{\int_0^{2\pi}\int_0^{3}\int_{r^2}^{36-3r^2} z\cdot r\,dz\,dr\,d\theta}{m}=\frac{4374\pi}{243\pi}=18$$

$\boxed{(\bar{x},\bar{y},\bar{z})=(0,0,18)}$

In review session I found center of mass of this region when the Q wants the region inside the lines