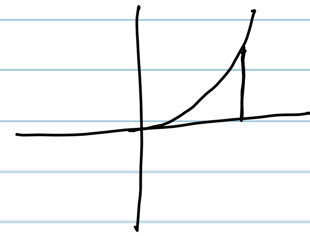
Example solutions

1)



$$m = \iint p(x,y)\,dA$$

$$= \int_0^1\int_0^{x^2} (x+y)\,dy\,dx = \int_0^1 xy + \frac{1}{2}y^2 \Big|_0^{x^2} dx$$

$$= \int_0^1 \left(x^3 + \frac{1}{2}x^4\right) dx = \frac{1}{4}x^4 + \frac{1}{10}x^5 \Big|_0^1 = \frac{1}{4} + \frac{1}{10}$$

$$m_x = \int_0^1\int_0^{x^2} y(x+y)\,dy\,dx = \int_0^1\int_0^{x^2} (xy+y^2)\,dy\,dx$$

$$= \int_0^1 \frac{1}{2}xy^2 + \frac{1}{3}y^3 \Big|_0^{x^2} dx = \int_0^1 \left(\frac{1}{2}x^5 + \frac{1}{3}x^6\right) dx$$

$$= \frac{1}{12}x^6 + \frac{1}{21}x^7 \Big|_0^1 = \frac{1}{12} + \frac{1}{21}$$

$$m_y = \int_0^1\int_0^{x^2} x(x+y)\,dy\,dx = \int_0^1\int_0^{x^2} (x^2+xy)\,dy\,dx$$

$$= \int_0^1 x^2y + \frac{1}{2}xy^2 \Big|_0^{x^2} = \int_0^1 x^4 + \frac{1}{2}x^5\,dx = \frac{1}{5} + \frac{1}{12}$$

$$\bar{x} = \frac{m_y}{m} = \frac{\frac{1}{5}+\frac{1}{12}}{\frac{1}{4}+\frac{1}{10}} \qquad \bar{y} = \frac{m_x}{m} = \frac{\frac{1}{12}+\frac{1}{21}}{\frac{1}{4}+\frac{1}{10}}$$

$$I_x = \int_0^1 \int_0^{x^2} y^2(x+y)\,dy\,dx = \frac{5}{72}$$

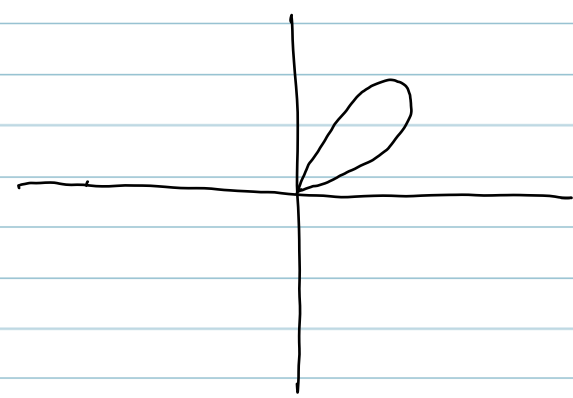
$$I_y = \int_0^1 \int_0^{x^2} x^2(x+y)\,dy\,dx = \frac{5}{21}$$

2/ If θ starts at 0 then r starts at $\sin(0)=0$.

Look for next value of θ where

$\sin(2\theta)=0$

$\theta = \frac{\pi}{2}$



so

$r=\sin(2\theta)\quad 0\le\theta\le\frac{\pi}{2}$

$$m = \iint p(x,y)\,dA \qquad p(x,y) = x^2+y^2 = r^2$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\sin(2\theta)} r^2\, r\,dr\,d\theta$$

$$m_x = \iint y\, \rho(x,y)\, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} r\sin\theta\, r^2\, r\, dr\, d\theta$$

$$m_y = \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} r\cos\theta\, r^2\, r\, dr\, d\theta$$

$$I_x = \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r^2 \sin^2\theta\, r^2\, r\, dr\, d\theta$$

$$I_y = \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r^2 \cos^2\theta\, r^2\, r\, dr\, d\theta$$

Example 3:

Spherical: $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^{2} 1\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Cylindrical: $\int_0^{2\pi} \int_0^{2} \int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta$

Rectangular: $\int \int \int_{\sqrt{1-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 1\, dz\, dx\, dy$

$x^2+y^2+z^2=1$

$r^2+z^2=1$

$z=\sqrt{1-r^2}$

$x^2+y^2+z^2=2^2$

$z=\sqrt{4-r^2}$

Example 3:

Spherical

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{2} 1\,\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

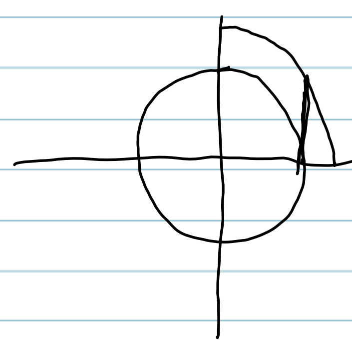
Cylindrical

$$\int_0^{2\pi}\int_0^{1}\int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} 1\cdot r \, dz \, dr \, d\theta + \int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{4-r^2}} 1\cdot r \, dz \, dr \, d\theta$$

Rectangular (Cartesian)

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{1-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx \; +$$

$$4\left[\int_0^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx + \int_1^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx\right]$$



4/

Let

$x-1=9u$ so $x=9u+1$

$y+1=16v$ $\quad y=16v-1$

S (region in u-v plave)

$$16(9u)^2+9(16v)^2=144$$

$$u^2+v^2=144$$

$$\iint_R 1\,dA=\iint_S 1\begin{vmatrix}9&0\\0&16\end{vmatrix}du\,dv$$

$$=\iint_S 144\,du\,dv \;\cancel{\pi}\; = \int_0^{2\pi}\int_0^1 144\cdot r\,dr\,d\theta$$

$u=r\cos\theta$

$v=r\sin\theta$ $\quad = \boxed{144\pi}$

Solution Example 5

Find f s.t. $\nabla f = F$

$$\frac{df}{dx} = yz + y + z + 1$$

$$f = \int (yz + y + z + 1)\,dx$$

$$= xyz + xy + xz + x + k(y,z)$$

$$\frac{df}{dy} = xz + x + z + 1$$

$$xz + x + \frac{dk}{dy} = xz + x + z + 1$$

$$\frac{dk}{dy} = z + 1 \qquad k = \int (z+1)\,dy$$

$$k = zy + y + h(z)$$

$$f = xyz + xy + xz + x + zy + y + h(z)$$

$$\frac{df}{dz} = xy + x + y + 1$$

$$xy + x + y + \frac{dh}{dz} = xy + x + y + 1$$

$$\frac{dh}{dz} = 1 \qquad h = \int 1\,dz \qquad h = z + C$$

$$f(x,y,z) = xyz + xy + xz + x + zy + y + z$$

$$\int_C F\cdot dr = \int_C \nabla f\cdot dr = \int_{(0,0,0)}^{(1,1,1)} \nabla f\cdot dr = f(1,1,1) - f(0,0,0)$$

$$= 7 - 0 = \boxed{7}$$

$$\int_C F\cdot dr = \int_0^1 (t^2t^3 + t^2 + t^3 + 1)\cdot 1 + (t\cdot t^3 + t + t^3 + 1)\cdot 2t + (tt^2 + t + t^2 + 1)\cdot 3t^2\, dt = 7$$

Solution Example 6:

$$\int_C 1\, ds = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt = 2\pi$$

$$\int_C \frac{1}{y}\, dx = \int_0^{2\pi} \frac{1}{\sin(t)} \cdot -\sin(t)\, dt = -2\pi$$