Exam III Review Sheet

Moments and Centers of Mass (in two dimensions)
p1017

$M_x = \iint_R y\,\rho(x,y)\,dA$ $\qquad \rho(x,y) = \text{density}$

$M_y = \iint_R x\,\rho(x,y)\,dA$

M_x: Moment about x-axis
M_y: Moment about y-axis
m: Mass of region

$m = \iint_R \rho(x,y)\,dA$

$\bar{x}$ = x coordinate center of mass
$\bar{y}$ = y coordinate center of mass

$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$

I_x: Moment of inertia about x-axis
I_y: Moment of inertia about y-axis

$I_x = \iint_R y^2\rho(x,y)\,dA \qquad I_y = \iint_R x^2\rho(x,y)\,dA$

Examples:

~~Suppose R is the region bounded by $x=0$, $y=e^x$, $x=1$ and $y=0$ with density $p(x,y)=$~~

1/ Suppose R is the region bounded by $y=x^2$, $y=0$ and $x=1$ and R has density $p(x,y)=x+y$. compute the center of mass for R and I_x, I_y.

2/ Suppose R is the region given by one loop of $r=\sin(2\theta)$. (with θ START AT

Set up but do not evaluate integrals for m, M_x, M_y, I_x, I_y if $p(x,y)=x^2+y^2$

Coordinate Systems

~~2D~~

Cartesian (x,y) $dA = dx\,dy$

Polar:

$x = r\cos\theta$ $dA = r\,dr\,d\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

~~3D~~

Cartesian $dA = dx\,dy\,dz$

Cylindrical:

$z = z$ $dA = r\,dr\,d\theta\,dz$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

Spherical:

$x = \rho\sin\phi\cos\theta$ $dA = \rho^2\sin\phi$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$x^2+y^2+z^2=\rho^2$

$\text{Volume} = \iiint_R 1\,dV$ $\text{Area} = \iint_R 1\,dA$

Example: 3/

~~Write the volume for the region between the spheres $\rho=1$ and $\rho=2$ above the xy plane $(z\geq 0)$ in spherical, cylindrical and cartesian coordinates~~

Write the volume for the region between the spheres $\rho=1$, $\rho=2$ above the xy plane, $(z\geq 0)$, in cylindrical, spherical and cartesian coordinates

Transformations

If T is a (one-to-one) transform (with non-zero Jacobian) mapping a region S in uv plane to R in xy plane then:

$$\iint_R f(x,y)\,dA = \cancel{\iint_S f(x(u,v),y(u,v))\,du\,dv}$$

$$= \iint_S f(x(u,v),y(u,v)) \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} du\,dv$$

Example: 4/

Use a transform to calculate the volume of the region R=

$16(x-1)^2+9(y+1)^2 \leq 144$

Hint: Try to make into circle in uv plane, then use polar coordinates

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Vector Fields

$F(x,y,z)=F_1(x,y,z)i+F_2(x,y,z)j+F_3(x,y,z)k$

(or as the book writes)

$= P(x,y,z)i+Q(x,y,z)j+R(x,y,z)k$

Vector field F is conservative if

$F=\nabla f$ for some function f

F is path independent iff

$$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$$ whenever C_1, C_2 have same start + end

F is conservative iff path independent iff $$\int_C F\cdot dr = 0$$ whenever C is a closed loop.

~~$\int_C f(x,y)\,ds =$ If $r(t)$ traces C and $r(t) = \langle r_1(t), r_2(t), r_3(t)\rangle$, $a \le t \le b$~~

~~$\int_C f(x,y,z)\,ds = \int_a^b f(r(t))\cdot\sqrt{\left(\frac{dr_1}{dt}\right)^2+\left(\frac{dr_2}{dt}\right)^2+\left(\frac{dr_3}{dt}\right)^2}\,dt$~~

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\cdot x'(t)\,dt$$

where $r(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$ traces C

same for dy, dz

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$\int_C F\cdot dr = \int_a^b F(x(t),y(t),z(t))\cdot\langle x'(t), y'(t), z'(t)\rangle\,dt$$

$$\int_C \nabla f\cdot dr = f(r(b)) - f(r(a))$$ ~~See end of example sols for example~~

Review Continued

$F = F_1 i + F_2 j$ is conservative on a nice (open, no holes) region if

$$\frac{dF_1}{dy} = \frac{dF_2}{dx}$$

$F = F_1 i + F_2 j + F_3 k$ is conservative on a nice region if

$$\frac{dF_1}{dy} = \frac{dF_2}{dx} \qquad \frac{dF_1}{dz} = \frac{dF_3}{dx} \qquad \frac{dF_2}{dz} = \frac{dF_3}{dy}$$

Example 5

Compute $\int_C F \cdot dr$ where C is given by $x=t, y=t^2, z=t^3, 0 \le t \le 1$

$F = \langle yz+y+z+1, xz+x+z+1, xy+x+y+1 \rangle$

Both by direct integration and by ~~checking if is~~ finding f s.t. $\nabla f = F$

Example 6:

Let C be given by

$r(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le 2\pi$

compute $\int_C 1 \, ds$, $\int_C \frac{1}{y} \, dx$