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Computably Enumerable Equivalence Relations

Abstract. We study computably enumerable equivalence relations (*ceers*) on \mathbb{N} and unravel a rich structural theory for a strong notion of reducibility among ceers.

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1. Introduction

DEFINITION 1.1. An equivalence relation R on \mathbb{N} is a *ceer* if R is a c.e. subset of \mathbb{N}^2 . Such equivalence relations are also called *positive*.

DEFINITION 1.2. For two ceers R_1 and R_2 , we say that R_1 is *m-reducible to* (or *many-one reducible to*) R_2 , denoted $R_1 \leq_m R_2$, if there is a computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that, for any $x, y \in \mathbb{N}$,

$$xR_1y \iff f(x)R_2f(y).$$

We say that R_1 is *1-reducible to* R_2 , denoted $R_1 \leq_1 R_2$, if there is a one-one computable function as above. Sometimes we omit the prefix and say that R_1 is reducible to R_2 , and denote $R_1 \leq R_2$. When we do this, the reducibility is meant to be many-one. We write $R_1 <_m R_2$ or $R_1 < R_2$ for the statement that $R_1 \leq R_2$ but $R_2 \not\leq R_1$. We write $R_1 \equiv R_2$ if $R_1 \leq R_2 \leq R_1$.

Note that this reducibility is a stronger notion than the ordinary reducibility among sets. Thus the following notion of “completeness” is stronger than the corresponding concept for sets.

DEFINITION 1.3. A ceer R is *universal* if $S \leq R$ for any ceer S .

Positive equivalence relations were first studied by Ershov [3], although he did not study the reducibility notion defined above. Ershov introduced the following notion of precompleteness.

DEFINITION 1.4. A ceer R is *precomplete* if R has infinitely many equivalence classes and for every partial computable function ψ there is a total computable function f such that for any $n \in \mathbb{N}$ with $\psi(n) \downarrow$, $f(n)R\psi(n)$.

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Visser [10] studied examples of ceers in logic, especially precomplete ones. However, it seems that Bernardi and Sorbi [2] were the first authors to have isolated the strong reducibility notion and studied universal ceers in the sense defined above. In particular, they proved that precomplete ceers must be universal. Lachlan [5] later showed that precomplete ceers are computably isomorphic to each other. He also considered another natural notion of completeness (called *e-complete*) and demonstrated nice properties and natural examples of e-complete ceers.

Ceers were also studied from a different perspective by Nies [7]. Some classes of ceers Nies considered will be redefined in this paper and their structure of reducibility will be investigated.

Examples of ceers in algebra, topology, logic and other areas of mathematics have been investigated under different disguises. For example, all word problems (not the identity problems) for semigroups and groups are in fact ceers. It is easily derivable from the proofs of the classical unsolvability results that there are semigroups and groups whose word problems are universal ceers. A non-trivial example was considered by Miller [6], who established that the isomorphism of finitely generated groups is a universal ceer.

Our primary goal in this paper is to develop a comprehensive theory for the structure of reducibility among ceers. Given the interesting circumstances that universal ceers are in some sense better understood, we will mostly focus on non-universal ceers in this paper. Our hope is to understand enough about non-universal ceers and the ways more complicated ceers can be built from simpler ones, so as to eventually clarify the structure of the strong reducibility.

The research work presented in this paper was to a large extent motivated by the theory of Borel reducibility among Borel (or other definable) equivalence relations, developed by Becker, KeCHRIS (e.g. [1]), Friedman, Stanley ([4]) and others. Although there are no direct connections between the two subjects, the general methodology we employ in the current research is much similar to the theory of Borel equivalence relations.

Let us briefly mention some speculations about the advantages to consider the strong reducibility notion. First of all, the most interesting applications of the theory of equivalence relations are often classification problems for various kinds of structures in mathematics. Furthermore, to classify a structure is usually meant to obtain a complete set of invariants for the structure. What we are investigating about ceers under the strong reducibility can then be viewed as an abstract framework for a study of the possibilities to effectively *compute* complete invariants. In this sense our reducibility is arguably more natural than the reducibility among sets.

Second, a ceer can be viewed as a c.e. decomposition of the universe. Hence the reducibility among ceers as equivalence relations generalizes various notion of simultaneous reduction for sequences of sets or reduction for “promise” problems. Also, natural non-universal ceers are easy to construct. This is in contrast with the situation in computability theory for sets, where sophisticated constructions are usually needed to ensure existence.

The rest of the paper is organized as follows. Section 2 contains some preliminaries about ceers. In sections 3–6 we consider ceers with simplest formations and explore the reducibility among them. In sections 7 and 8 we defined two notion of jump operations for ceers. As we have mentioned above, more complicated ceers can thus be formed and more interesting properties can be obtained. In sections 8 and 9 we clarify the structure of reducibility among the ceers constructed in earlier sections. Section 10 contains a summary of the theory and some open problems to consider in future researches.

2. Indices for ceers

Our basic notation about partial computable functions and c.e. sets follows that of [9]. The partial computable function with index e is denoted by φ_e . The c.e. set with index e is denoted by W_e . If ψ is a partial computable function and $x \in \mathbb{N}$, then the domain of ψ is denoted by $\text{dom } \psi$; $\psi(x) \downarrow$ indicates that $x \in \text{dom } \psi$ and $\psi(x) \uparrow$ otherwise. For a partial computable function ψ and $n \in \mathbb{N}$, ψ^n denotes the partial computable function obtained by n iterated compositions of ψ . This can be defined precisely by induction on n : $\psi^0(x) = x$ and $\psi^{n+1}(x) = \psi(\psi^n(x))$ for any $x \in \mathbb{N}$.

Throughout the paper we adopt the following notation. Denote $K = \{x \in \mathbb{N} \mid \varphi_x(x) \downarrow\}$ and $\overline{K} = \mathbb{N} \setminus K$. For $i \in \mathbb{N}$, let $K_i = \{x \in K \mid \varphi_x(x) \downarrow = i\}$. We use the notation $\langle \cdot, \cdot \rangle$ to denote a fixed computable bijection between \mathbb{N}^2 and \mathbb{N} and call it the *coding function*. Coding functions of arbitrary arities are similarly represented.

Next we fix some notation and note some basic facts about ceers.

For a ceer R on \mathbb{N} and $x \in \mathbb{N}$, denote by $[x]_R$ the R -equivalence class containing x and by $|[x]_R|$ the cardinality of $[x]_R$.

The set of all ceers is computably enumerable. Let R_e be the equivalence relation generated by the set $\{(x, y) \mid \langle x, y \rangle \in W_e\}$. Then R_e is a ceer and all ceers appear in this enumeration as some R_e . We will regard this as the canonical enumeration for all ceers and say that e is a *canonical c.e. index* for the relation R_e .

Let R_∞ be defined by

$$\langle x, z \rangle R_\infty \langle y, z \rangle \iff x R_z y$$

for $x, y, z \in \mathbb{N}$. (It is to be understood in this definition, as well as in all definitions of a similar form in the sequel, that no other pair of distinct numbers is in the relation.) It is obvious that R_∞ is a universal ceer.

There are other procedures which can generate all ceers. The following definition is due to Ershov.

DEFINITION 2.1. For each partial computable function f , define the ceer η_f by

$$x \eta_f y \iff \exists n, m (f^n(x) \downarrow = f^m(y) \downarrow), \text{ for } x, y \in \mathbb{N}.$$

Then η_f is the ceer generated by the graph of f . If e is an index for f , then we also write η_f as η_e , and we call e the *iterative c.e. index* for η_e .

Ershov [3] showed that every ceer has an iterative c.e. index. We note below that the two indexing systems are essentially the same.

THEOREM 2.2. *There is a computable isomorphism ρ such that $\eta_e = R_{\rho(e)}$.*

PROOF. A proof of Ershov's result indicates, in fact it is not hard to see directly, that there is a one-one computable function $f(n, e)$ such that $R_e = \eta_{f(n, e)}$ for any $n, e \in \mathbb{N}$. Similarly, there is a one-one computable function $g(n, e)$ such that $\eta_e = R_{g(n, e)}$ for any $n, e \in \mathbb{N}$. With this padding property it is then routine to define a computable isomorphism ρ by a back-and-forth construction as follows.

Let $\rho(0) = f(0, 0)$. If $f(0, 0) = 0$ then $\rho^{-1}(0)$ is already defined. Otherwise, we define $\rho^{-1}(0)$ to be $g(k, 0)$ where k is the least such that $g(k, 0) \neq 0$. In general suppose both ρ and ρ^{-1} are defined for $i < n$. If $n \in \{\rho^{-1}(0), \dots, \rho^{-1}(n-1)\}$ then $\rho(n)$ is already defined. Otherwise let $\rho(n)$ be $f(k, n)$ where k is the least such that $f(k, n) \notin \{\rho(0), \dots, \rho(n-1)\}$. Then in a similar way define $\rho^{-1}(n)$. The resulting ρ is one-one, onto and computable. ■

Ceers can also be generated by computable actions of countable computable groups on \mathbb{N} . When the ceer has no finite equivalence classes, the group can be taken to be the additive group of \mathbb{Z} .

It will be useful to note that the reducibility among ceers is a Σ_3 relation. Specifically, the index set $\{(e, i) \mid R_e \leq R_i\}$ is Σ_3 . This is by a straightforward computation. It follows that the set $\{(e, i) \mid R_e \equiv R_i\}$ is also Σ_3 . In the subsequent sections we will show that both reducibility and bi-reducibility among ceers are Σ_3 -complete.

In the subsequent sections we will consider various classes of ceers, and we will adopt the following terminology.

DEFINITION 2.3. Let \mathcal{C} be a non-empty class of ceers. A ceer R is said to be *essentially* in \mathcal{C} if $R \leq S$ for some $S \in \mathcal{C}$. R is *universal* for \mathcal{C} if $R \in \mathcal{C}$ and $S \leq R$ for any $S \in \mathcal{C}$. If \mathcal{C} has a universal relation in it, then R is *essentially universal* for \mathcal{C} if $R \equiv S$ for some S universal for \mathcal{C} .

Much of our investigation is focused on finding and describing universal relations for different classes of ceers.

3. Computable equivalence relations and partial classifiability

Computable equivalence relations are the simplest ones among ceers with respect to reducibility.

DEFINITION 3.1. For each $n \in \mathbb{N}$ and $n > 0$, let $\text{id}(n)$ denote the following equivalence relation on \mathbb{N} with n equivalence classes:

$$x \text{id}(n)y \iff x \equiv y \pmod{n}, \text{ for } x, y \in \mathbb{N}.$$

Let $\text{id}(\mathbb{N})$ or ω be the identity relation on \mathbb{N} :

$$x\omega y \iff x = y, \text{ for } x, y \in \mathbb{N}.$$

DEFINITION 3.2. Let R be a ceer on \mathbb{N} . A subset $T \subseteq \mathbb{N}$ is called a *transversal* for R if T meets each R -equivalence class in exactly one element.

Then we have the following complete characterization and classification of computable equivalence relations.

PROPOSITION 3.3.

- (1) For each $n \in \mathbb{N}$ and $n > 0$, $\text{id}(n)$ is computable; and ω is computable.
- (2) $\text{id}(1) < \text{id}(2) < \dots < \text{id}(n) < \dots < \omega$.
- (3) If $R < \omega$, then there is some $n \in \mathbb{N}$ such that $R \equiv \text{id}(n)$.

PROOF. (1) and (2) are obvious. For (3), suppose $R < \omega$, then it follows that R is computable. If R has infinitely many equivalence classes, then $\omega \leq R$ via the following computable function f defined by primitive recursion:

$$\begin{aligned} f(0) &= 0 \\ f(x+1) &= \mu y (y > f(x) \wedge \forall z \leq x \neg (yRf(z))). \end{aligned}$$

It follows that R has only finitely many equivalence classes. Let $n \in \mathbb{N}$ be the number of equivalence classes of R . Then it is easy to see that $R \equiv \text{id}(n)$. ■

PROPOSITION 3.4. *The following are equivalent for any ceer R :*

- (i) R is computable;
- (ii) $R \leq \omega$;
- (iii) R has an c.e. transversal;
- (iv) R has a computable transversal;
- (v) $T(R) =_{\text{def}} \{\min(C) \mid C \text{ is an } R\text{-equivalence class}\}$ is computable.

PROOF. Note that (i) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (iii) and (ii) \Rightarrow (i) are obvious. We are only left to prove (iii) \Rightarrow (ii). Suppose T is an c.e. transversal for R . Consider the function $g: \mathbb{N} \rightarrow T$ defined by letting $g(x)$ be the unique element $y \in T$ with xRy . Then g is total because T is a transversal. It is obvious that the graph of g is c.e., hence g is computable. But then g is a reduction from R to ω . ■

The proposition implies that computable ceers are exactly the kind of ceers for which natural numbers can be effectively assigned to equivalence classes as complete invariants. Such equivalence relations should be considered classifiable in an ideal sense. Historically people have also considered the possibility of assigning an infinite system of natural numbers as complete invariants (e.g. [8]), in the following we remark that infinite systems of invariants do not help.

PROPOSITION 3.5. *Let R be a non-computable ceer. Then there is no computable function $f(m, n)$ such that, for any $x, y \in \mathbb{N}$,*

$$xRy \iff \forall m (f(m, x) = f(m, y)).$$

PROOF. The existence of such an f would imply that R is co-c.e.; since R is c.e. it would follow that R is computable, contradiction. ■

Next we formulate a notion of partial classifiability for ceers.

DEFINITION 3.6. For each partial computable function ψ define ceer P_ψ by

$$xP_\psi y \iff x = y \vee \psi(x) \downarrow = \psi(y) \downarrow, \text{ for } x, y \in \mathbb{N}.$$

Since ψ can be viewed as a function partially assigning complete invariants for (at least) the non-trivial equivalence classes, we call ceers of the form P_ψ *partially classifiable* or in abbreviated form, *PC*.

It turns out that there is a universal PC relation.

DEFINITION 3.7. The ceer H (standing for *halting equivalence*) is defined for $x, y \in \mathbb{N}$ by

$$xHy \iff x = y \vee \varphi_x(x) \downarrow = \varphi_y(y) \downarrow.$$

It is immediate from the definition that H is PC.

PROPOSITION 3.8. *H is universal for PC relations, i.e., for any PC relation R , $R \leq H$.*

PROOF. Suppose R is PC, as witnessed by partial computable function ψ . Let f be one-one computable function obtained by *s-m-n* Theorem such that $\varphi_{f(x)}(y) = \psi(x)$. Then f witnesses that $R \leq H$. ■

The following simple observation will be useful in § 7.

PROPOSITION 3.9. *Let E be a ceer without computable classes. Let R be a PC-relation. Then $E \not\leq R$.*

PROOF. Suppose f is a computable function witnessing $E \leq R$. Suppose ψ is a partial computable function with $R = P_\psi$. Then we claim that, for every $x \in \mathbb{N}$, $\psi(f(x))$ is defined. Otherwise, we have that $[f(x)]_R$ is a singleton containing only $f(x)$, thus by the assumption that f is a reduction, we would have that $[x]_E$ is computable. Therefore we in fact have that $\psi \circ f$ is a total computable function; moreover, $\psi \circ f$ witnesses that $E \leq \omega$, again implying that every E -class is computable, contradiction. ■

4. Finite dimensional ceers

DEFINITION 4.1. Let $n \in \mathbb{N}$ and $n > 0$. A ceer R is *n-dimensional* if there are pairwise disjoint c.e. sets A_1, \dots, A_n such that

$$xRy \iff x = y \vee \exists i \leq n (x, y \in A_i).$$

We denote the above relation by R_{A_1, \dots, A_n} . A ceer is *finite dimensional* if it is n -dimensional for some $n \in \mathbb{N}$.

We have the following easy facts about 1-dimensional ceers.

PROPOSITION 4.2. *Let A, B be non-empty c.e. sets.*

- (1) R_A is computable iff A is computable.
- (2) $A \leq_1 B$ iff $R_A \leq_1 R_B$.
- (3) If $R_A \leq R_B$, then $A \leq_m B$.

PROOF. For (1) one direction follows immediately from the definition. Conversely, suppose R_A is computable, then it has a computable transversal T . Note that $T \cap A$ is a singleton, therefore $T \setminus A$ is computable. But $T \setminus A$ is just the complement of A , it follows that A is computable. (2) and (3) are obvious. ■

The proposition indicates that the reducibility among 1-dimensional relations is very similar to that among c.e. sets. In fact, the proof of (1) shows that R_A (as a subset of \mathbb{N}^2) is in the same Turing degree as A . Because of the transparency of the reducibility among 1-dimensional ceers modulo that among the c.e. sets, we consider the 1-dimensional ceers completely classified.

PROPOSITION 4.3. *Let $n \in \mathbb{N}$ and $n > 0$.*

- (1) *If R is an n -dimensional ceer, then $R \leq_1 R_{K_0, \dots, K_{n-1}}$.*
- (2) *$R_{K_0, \dots, K_{n-1}} < R_{K_0, \dots, K_n}$.*
- (3) *Let $k \leq n$ and $R = R_{A_1, \dots, A_n}$. Then R is essentially k -dimensional iff at most k of the sets A_1, \dots, A_n are non-computable.*

PROOF. First consider (1). Let A_0, \dots, A_{n-1} be disjoint c.e. sets. We are to define a reduction to witness $R_{A_0, \dots, A_{n-1}} \leq R_{K_0, \dots, K_{n-1}}$. First define a partial computable function F as follows.

$$F(e, x, y) = \begin{cases} i & \text{if } i < n, x \in A_i \text{ and } y = \varphi_e(x) \\ \uparrow & \text{otherwise} \end{cases}$$

Let $s(e, x)$ be a one-one total computable function with $F(e, x, y) = \varphi_{s(e, x)}(y)$. Then by the fixed point theorem, let e_0 be such that $\varphi_{e_0}(x) = s(e_0, x)$, for all x . Finally let $f(x) = \varphi_{e_0}(x)$. Then f is one-one computable. We verify that f is the required reduction. If $x \in A_i$ for some $i < n$, then $F(e_0, x, f(x)) = i$ by definition of F , which means $\varphi_{s(e_0, x)}(f(x)) = \varphi_{f(x)}(f(x)) = i$, or $f(x) \in K_i$. If $x \notin A_i$ for any $i < n$, then by the definition of F , $F(e_0, x, f(x)) \uparrow$, which means $f(x) \notin K_i$ for any $i < n$.

For (2), note first that it follows from (1) that $R_{K_0, \dots, K_{n-1}} \leq R_{K_0, \dots, K_n}$. To see that $R_{K_0, \dots, K_n} \not\leq R_{K_0, \dots, K_{n-1}}$, just note that for any reduction function h there is some $i \leq n$ such that h sends the class K_i to a single element. This implies that K_i is computable, a contradiction.

(3) can be shown by a similar argument as in the last paragraph. ■

The proposition establishes a hierarchy for finite dimensional ceers with a universal relation in each level.

At this point it is natural to ask whether the identity relation is reducible to all non-computable ceers. We prove that this is not the case. Furthermore, we completely characterize those finite dimensional ceers to which the identity relation is reducible. Recall the definition of simple sets by Post.

DEFINITION 4.4. An c.e. set $A \subseteq \mathbb{N}$ is *simple* if

- (a) the complement of A is infinite, and
- (b) for any infinite c.e. set $B \subseteq \mathbb{N}$, $A \cap B \neq \emptyset$.

It is well known that simple sets exist and that none of them is m -complete among the c.e. sets.

PROPOSITION 4.5. *Let A_1, \dots, A_n be disjoint c.e. sets the complement of whose union is infinite. Then*

$$\omega \leq R_{A_1, \dots, A_n} \iff A_1 \cup \dots \cup A_n \text{ is not simple.}$$

PROOF. Let $A = \bigcup_{i \leq n} A_i$. Suppose f witness the reduction $\omega \leq R_{A_1, \dots, A_n}$. Let S be the range of f . Then $S \cap A$ contains at most n elements. It follows that $S \setminus A$ is an infinite c.e. set. Therefore A is not simple. Conversely, if A is not simple, then its complement contains an infinite c.e. set. Let f be a computable function enumerating this set. Then f is a reduction from ω to R_{A_1, \dots, A_n} . ■

Next let us remark that all finite dimensional ceers are PC.

PROPOSITION 4.6. *All finite dimensional ceers are PC.*

PROOF. Let A_1, \dots, A_n be disjoint c.e. sets. Then R_{A_1, \dots, A_n} is PC as witnessed by the following partial computable function ψ , defined by

$$\psi(x) = \begin{cases} i & \text{if } x \in A_i \\ \uparrow & \text{otherwise} \end{cases} \quad \blacksquare$$

Thus the halting equivalence H is an upper bound for all finite dimensional ceers in the reducibility order. In fact, the definitions suggest that H is quite a natural limit for all R_{K_0, \dots, K_n} . Nevertheless, it is not clear whether H is a *least* upper bound.

We are now ready to show that reducibility and bi-reducibility among ceers are Σ_3 -complete.

PROPOSITION 4.7. *The set $\{e \mid R_e \leq \omega\}$ is Σ_3 -complete.*

PROOF. It is well known that the set $\text{Comp} = \{i \mid W_i \text{ is computable}\}$ is Σ_3 -complete (see [9] Theorem 14-XVI). For each $i \in \mathbb{N}$, consider the one-dimensional ceer R_{W_i} . It is obvious that $R_{W_i} \leq \omega$ iff R_{W_i} is computable iff W_i is computable. Moreover, it is easy to see that there is a computable function f such that $R_{f(i)} = R_{W_i}$. Therefore Comp is reducible to the set in question. ■

This immediately gives the Σ_3 -completeness of reducibility and bi-reducibility.

COROLLARY 4.8. *The reducibility and bi-reducibility among ceers is Σ_3 -complete.*

PROOF. The assertion about reducibility follows trivially from the above proposition. For bi-reducibility consider, for each $i \in \mathbb{N}$, the one-dimensional ceer R_{2W_i} , where $2W_i = \{2x \mid x \in W_i\}$. Then $\omega \leq R_{2W_i}$ for every $i \in \mathbb{N}$. Moreover, $R_{2W_i} \leq \omega$ iff $2W_i$ is computable iff W_i is computable. We thus have that the set $\{e \mid R_e \equiv \omega\}$ is Σ_3 -complete. ■

Next we consider the complexity of various other sets arising from the reducibility of ceers.

LEMMA 4.9. *The index set $\{e \mid \exists i (W_i \text{ is infinite and } W_i \cap W_e = \emptyset)\}$ is Σ_3 -complete.*

PROOF. It is well known that the set $\{e \mid W_e \text{ is simple}\}$ is Π_3 -complete. A proof of this can be found in [9] Exercise 14–31, which uses eventually a theorem of Dekker ([9] Theorem 9-XVI). Note that Dekker’s proof effectively shows that there is a computable function f such that, for any x , $W_{f(x)}$ is co-infinite, and W_x is not computable iff $W_{f(x)}$ is simple (in fact hypersimple). This reduction function f witnesses that Comp is reducible to our set in question. ■

PROPOSITION 4.10. *The set $\{e \mid \omega \leq R_e\}$ is Σ_3 -complete.*

PROOF. This now follows easily from Proposition 4.5 and Lemma 4.9. ■

By similar proofs we obtain the following results.

PROPOSITION 4.11. *The following index sets are Σ_3 -complete:*

- (1) $\{e \mid R_e \leq R_A\}$ and $\{e \mid R_e \equiv R_A\}$, where $A \subseteq \mathbb{N}$ is non-empty, c.e., not simple and has infinite complement.
- (2) $\{e \mid R_e \text{ is essentially } n\text{-dimensional}\}$, where $n > 0$.
- (3) $\{e \mid R_e \text{ is essentially universal } n\text{-dimensional}\}$, where $n > 0$.
- (4) $\{e \mid R_e \text{ is essentially finite dimensional}\}$.

PROOF. Let us sketch the proof for (1); the other parts are similar. Let A be as given and C be a computable infinite subset of $\mathbb{N} \setminus A$. Fix a computable bijection θ from \mathbb{N} onto C . For each e consider the 2-dimensional ceer $R_{A, \theta(W_e)}$. By Proposition 4.3 it is easy to see that $R_{A, \theta(W_e)} \leq R_A$ iff W_e is computable, and when this happens we also have $R_{A, \theta(W_e)} \equiv R_A$. Similar to the proof of Proposition 4.7 we can then construct reductions from Comp to the index sets $\{e \mid R_e \leq R_A\}$ and $\{e \mid R_e \equiv R_A\}$. ■

5. FC relations and CC relations

Since ceers with only finitely many equivalence classes are not very interesting in terms of reducibility, we call such relations *trivial*.

DEFINITION 5.1. A ceer R is *FC* (standing for *finite classes*) if all of its equivalence classes are finite. R is *CC* (for *computable classes*) if all of its equivalence classes are computable.

Let us summarize some basic facts.

PROPOSITION 5.2.

- (1) All FC relations are CC.
- (2) If $R_1 \leq R_2$ and R_2 is CC, then R_1 is CC.
- (3) If R is a PC-relation whose every equivalence class is non-trivial, then R is computable. Therefore, there are CC relations which are not PC. In fact, for each non-computable PC-relation R there is $S \equiv R$ such that S is not PC.
- (4) For each FC relation R there is $S \equiv R$ such that S is not FC.

PROOF. (1), (2) and the first part of (3) are obvious. For the second part of (3) and (4) define S by

$$\langle x, u \rangle S \langle y, v \rangle \iff x R y,$$

then every equivalence class of S is infinite, and it is easy to see that $S \equiv R$. ■

We have the following non-reducibility result between finite dimensional relations and CC relations.

PROPOSITION 5.3. Let R be a non-computable finite dimensional ceer and S be a non-computable CC relation. Then $R \not\leq S$ and $S \not\leq R$.

PROOF. Note the general fact: if $R_1 \leq R_2$ via f and $[y]_{R_2}$ is computable, then the preimage set $C = \{x \mid f(x) R_2 y\}$ is also computable. In fact, either C is empty or C is a single R_1 -equivalence class. Moreover $C \leq_m [y]_{R_2}$ via f . Let R and S be given as in the hypothesis. Then $R \not\leq S$ is an immediate consequence of the above fact. For $S \not\leq R$, suppose $R = R_{A_1, \dots, A_n}$ where A_1, \dots, A_n are disjoint c.e. non-computable. Assume that f witnesses $S \leq R$ and suppose $B = \{f(x) \mid x \in \mathbb{N}\}$. Let $A = \bigcup_{i \leq n} A_i$. If $B \cap A = \emptyset$, then f is in fact a witness for $S \leq \omega$, contradicting the hypothesis that S is non-computable. Otherwise, let C_1, \dots, C_k ($k \leq n$) be the S -equivalence classes

with $f(x) \in A_i$ for $i \leq k$ and $x \in C_i$. Choose arbitrary elements $y_i \in A_i$ for $i \leq k$ and define

$$g(x) = \begin{cases} y_i & \text{if } i \leq k \text{ and } x \in C_i \\ f(x) & \text{otherwise} \end{cases}$$

Then g witnesses that $S \leq \omega$, again contradicting the hypothesis that S is non-computable. \blacksquare

We next consider the question of when $\omega \leq R$ for an FC relation R .

DEFINITION 5.4. For an c.e. set $A \subseteq \mathbb{N}$, let F_A be the equivalence relation defined by

$$xF_Ay \iff x = y \vee \forall z(\min(x, y) \leq z \leq \max(x, y) \rightarrow z \in A), \text{ for } x, y \in \mathbb{N}.$$

Then F_A is a ceer.

If A is an c.e. set whose complement is infinite, then F_A is an FC relation. In analogy with the result of preceding section we have the following proposition. Let us recall the definition of hypersimple sets, also due to Post.

DEFINITION 5.5.

- (a) Let $A \subseteq \mathbb{N}$ be an infinite set and f a total function on \mathbb{N} . We say that f *majorizes* A if, letting z_0, z_1, \dots be the enumeration of A in strictly increasing order, we have that $(\forall n)f(n) \geq z_n$.
- (b) An c.e. set $A \subseteq \mathbb{N}$ is *hypersimple* if the complement of A is infinite and there is no computable f such that f majorizes $\mathbb{N} \setminus A$.

PROPOSITION 5.6. *Let $A \subseteq \mathbb{N}$ be an c.e. set whose complement is infinite. Then*

$$\omega \leq F_A \iff A \text{ is not hypersimple.}$$

PROOF. Suppose f witnesses that $\omega \leq F_A$. Then define by recursion a function g as follows.

$$\begin{aligned} g(0) &= f(0), \\ g(n+1) &= f(x), \text{ where } x \text{ is the least with } f(x) > g(n). \end{aligned}$$

Since the complement of A is infinite, g is total computable. Letting $h(n) = g(n+1)$, then h majorizes $\mathbb{N} \setminus A$, therefore A is not hypersimple. Conversely, if A is not hypersimple, let h be a computable function majorizing $\mathbb{N} \setminus A$ and let z_0, z_1, \dots be an enumeration of $\mathbb{N} \setminus A$ in strictly increasing order.

We define a computable function f enumerating an infinite set of pairwise inequivalent elements of F_A , then f witnesses that $\omega \leq F_A$. Let

$$\begin{aligned} f(0) &= 0, \\ f(n+1) &= h(f(n)). \end{aligned}$$

Then $f(n) \leq z_{f(n)} < h(f(n)) = f(n+1)$. Since $z_{f(n)} \notin A$, by definition of F_A we have that $f(n)$ and $f(n+1)$ are inequivalent. ■

The following results concern the complexity of being FC, essentially FC and CC.

LEMMA 5.7. *There is a computable function f such that, for any $x \in \mathbb{N}$,*

$$\begin{aligned} W_x \text{ is co-finite} &\Rightarrow R_{f(x)} \text{ is not CC, and} \\ W_x \text{ is co-infinite} &\Rightarrow R_{f(x)} \text{ is FC.} \end{aligned}$$

PROOF. Given an c.e. set W define a ceer R^W by

$$xRy \iff x = y \vee (\forall z(\min(x, y) \leq z \leq \max(x, y) \rightarrow z \in W) \wedge x, y \in K).$$

Then if W is co-finite R is a finite modification of R_K , hence R is not CC. If W is co-infinite then R is obviously FC. ■

PROPOSITION 5.8.

- (1) *The set $\{e \mid R_e \text{ is FC}\}$ is Π_3 -complete.*
- (2) *The set $\{e \mid R_e \text{ is essentially FC}\}$ is Π_3 -hard.*

PROOF. These follow from the preceding lemma and some additional observations. First of all, note that the set $\{e \mid W_e \text{ is co-finite}\}$ is known to be Σ_3 -complete. Then, for (1) just note that being FC is Π_3 . For (2) note that if R is not CC, then R is not essentially FC. ■

LEMMA 5.9. *The set $\{e \mid \forall x \{y \mid \langle x, y \rangle \in W_e\} \text{ is computable}\}$ is Π_4 -complete.*

PROOF. Denote the set by X . X is obviously Π_4 . To see that it is Π_4 -complete, let A be an arbitrary Π_4 set and let $B \in \Sigma_3$ be such that

$$x \in A \iff \forall y(\langle x, y \rangle \in B), \text{ for all } x.$$

Let f be a reduction function from B to Comp. Define a computable function g such that, for any $e, x, y \in \mathbb{N}$,

$$\langle y, x \rangle \in W_{g(e)} \iff x \in W_{f(\langle e, y \rangle)}.$$

Then for any $e \in \mathbb{N}$,

$$\begin{aligned}
e \in A &\Leftrightarrow \forall y (\langle e, y \rangle \in B) \\
&\Leftrightarrow \forall y (f(\langle e, y \rangle) \in \text{Comp}) \\
&\Leftrightarrow \forall y (W_{f(\langle e, y \rangle)} \text{ is computable}) \\
&\Leftrightarrow \forall y (\text{the set } \{x \mid \langle y, x \rangle \in W_{g(e)}\} \text{ is computable}) \\
&\Leftrightarrow g(e) \in X
\end{aligned}$$

This shows that $A \leq_m X$. Since A is arbitrary, X is Π_4 -complete. \blacksquare

PROPOSITION 5.10. *The set $\{e \mid R_e \text{ is CC}\}$ is Π_4 -complete.*

PROOF. A straightforward computation shows that it is Π_4 . To see that it is Π_4 -complete, for any c.e. set W define a ceer R^W by

$$\langle x, y \rangle R^W \langle x, z \rangle \iff y = z \vee (\langle x, y \rangle, \langle x, z \rangle \in W)$$

for any $x, y, z \in \mathbb{N}$. Let X be the index set in the preceding lemma. Then $e \in X$ iff the relation R^{W_e} is CC. \blacksquare

COROLLARY 5.11.

- (1) *There is no universal FC relation.*
- (2) *There is no universal CC relation.*
- (3) *There is an CC relation which is not essentially FC.*

PROOF. For (1) and (2) note that if there were a universal FC or CC relation, then being essentially FC or being CC would be Σ_3 , contradicting Proposition 5.8(2) and Proposition 5.10. For (3), if every CC relation were essentially FC, then the sets $\{e \mid R_e \text{ is CC}\}$ and $\{e \mid R_e \text{ is essentially FC}\}$ would be the same. But this is impossible since the former is Π_4 -complete and the latter is Σ_4 . \blacksquare

Part (3) of this corollary asserts the abstract existence of some ceers of which we do not yet have any constructed example.

6. Bounded relations

DEFINITION 6.1. For $k \in \mathbb{N}$ and $k > 0$, we say that R is *k-bounded* if every R -equivalence class contains at most k elements. R is *bounded* if it is k -bounded for some $k \in \mathbb{N}$.

The following theorem shows that $\omega \leq R$ for all bounded relations R but there is no uniform effective way to find the reduction.

THEOREM 6.2.

- (1) *If R is a bounded relation, then $\omega \leq R$.*
- (2) *Let $k \in \mathbb{N}$ and $k > 1$. There is no partial computable function $\rho(e, x)$ such that, whenever R_e is a k -bounded ceer then $\rho(e, x) \downarrow$, for all $x \in \mathbb{N}$ and $\rho(e, \cdot)$ witnesses the reduction from ω to R_e .*

PROOF. (1) Suppose R is k -bounded. For each $i \leq k$, let N_i be the number of R -equivalence classes containing exactly i many elements. Since $\sum_{i \leq k} N_i = \infty$, there is a biggest $l \leq k$ such that $N_l = \infty$. Let F be the set of all elements whose R -equivalence class has more than l elements. Then F is a finite set. Now let Γ be an enumeration of the pairs of elements in R . We define a computable reduction from ω to R by listing an infinite set of pairwise non- R -equivalent elements by stages. At stage n suppose a finite set A_n of pairwise inequivalent elements has been listed. Then carry out the n -th stage of Γ and enumerate a pair (x, y) . If $x \notin F$ and there have been l elements enumerated in $[x]_R$ by Γ , then we know that these l elements form exactly $[x]_R$. In this situation list the smallest element of $[x]_R$, provided that it has not been listed before, and go on to the next stage. In the situation that the above conditions are not fulfilled, do nothing and go on to the next stage. From our assumption that $N_l = \infty$ it follows that this procedure would produce infinitely many elements, and from the construction they are pairwise inequivalent. This finishes the proof for (1).

For (2), it suffices to show the conclusion for $k = 2$. Toward a contradiction, assume there is such a partial computable function ρ . Let s be a computable function such that for any e the ceer $R_{s(e)}$ is given by

$$xR_{s(e)}y \iff x = y \vee (x = \rho(e, 0) \downarrow \wedge y = \rho(e, 1) \downarrow) \\ \vee (x = \rho(e, 1) \downarrow \wedge y = \rho(e, 0) \downarrow),$$

for $x, y \in \mathbb{N}$. It is obvious that $R_{s(e)}$ is 2-bounded for any e . Now by the fixed point theorem there is some e_0 such that $W_{e_0} = W_{s(e_0)}$. For this e_0 we have that $R_{e_0} = R_{s(e_0)}$. Since $R_{s(e)}$ is always 2-bounded, it follows that R_{e_0} is 2-bounded and that for all x , $\rho(e_0, x) \downarrow$. But then by the definition of $R_{s(e_0)}$ we have that $\rho(e_0, 0)R_{s(e_0)}\rho(e_0, 1)$, which implies that $\rho(e_0, 0)R_{e_0}\rho(e_0, 1)$, which is a contradiction to our assumption that $\rho(e_0, \cdot)$ witnesses a reduction from ω to R_{e_0} . ■

Fix $k > 1$. Note that there is a computable enumeration of all k -bounded relations. A canonical enumeration can be given as follows. For an index

e , let R_e be the e -th ceer in the canonical enumeration of all ceers. We modify R_e by omitting any pair in its enumeration which would give rise to an equivalence class of size $> k$. Denote this resulting k -bounded relation by B_e^k . Then $\{B_e^k\}_{e \in \omega}$ enumerates exactly all k -bounded relations.

For each $k > 1$, define a ceer B_∞^k by

$$\langle x, z \rangle B_\infty^k \langle y, z \rangle \iff x B_z^k y, \text{ for } x, y, z \in \mathbb{N}.$$

Then B_∞^k is k -bounded and is in fact universal for all k -bounded relations, i.e., if R is an arbitrary k -bounded relation, then $R \leq B_\infty^k$.

The following theorem establishes a hierarchy for bounded relations in analogy with the hierarchy of finite dimensional relations.

THEOREM 6.3. (Bounded Hierarchy Theorem) *For any $k > 0$, $B_\infty^k < B_\infty^{k+1}$.*

PROOF. It suffices to show that there is a $k + 1$ -bounded relation which is not essentially k -bounded. For this, let A be a non-computable c.e. set and define a ceer R by

$$\langle x, i \rangle R \langle x, j \rangle \iff i = j \vee (i, j \leq k \wedge x \in A),$$

for $x, i, j \in \mathbb{N}$. Then R is $k + 1$ -bounded.

Now assume R is essentially k -bounded, i.e., there are computable function f and k -bounded relation S such that $R \leq S$ via f . Then we have

$$x \in A \iff \exists i, j \leq k (i \neq j \wedge f(\langle x, i \rangle) = f(\langle x, j \rangle)).$$

In fact, if $x \in A$, then $[\langle x, 0 \rangle]_R$ has $k + 1$ elements; and since f is a reduction, it must map the $k + 1$ elements into some single S -class, which only has at most k -elements. On the other hand, if $x \notin A$, then the elements $\langle x, i \rangle$ are pairwise R -inequivalent; again by that f is a reduction, it follows that the statement on the right hand side cannot happen. Now the statement gives a computable definition for A , a contradiction to our hypothesis that A is non-computable. \blacksquare

The idea of the above proof can be used to show that there are FC relations which are not essentially bounded.

PROPOSITION 6.4. *There are FC relations which are not essentially bounded.*

PROOF. Again let A be a non-computable c.e. set and define a ceer R by a slight modification to the previous definition, as follows. Let

$$\langle x, i \rangle R \langle x, j \rangle \iff i = j \vee (i, j \leq x \wedge x \in A),$$

for $x, i, j \in \mathbb{N}$. It is easy to see that R is FC.

Now assume R is essentially k -bounded and let f be a computable reduction function witnessing this. Then by the same argument as before, we have that for $x > k$,

$$x \in A \iff \exists i, j \leq x (i \neq j \wedge f(\langle x, i \rangle) = f(\langle x, j \rangle)),$$

which gives again a computable definition for A , a contradiction. \blacksquare

This proposition is also a corollary of Proposition 5.8 (1) and the fact that being essentially bounded is Σ_3 . But of course, the construction here makes the proof more favorable than the abstract comparison of the complexities. Results in § 8 will show that if A is hypersimple, then the FC relation F_A defined in Definition 5.4 is not essentially bounded. This provides another proof of Proposition 6.4.

Similar to Proposition 4.11, we can characterize the complexity of various index sets related to boundedness.

PROPOSITION 6.5. *The following index sets are Σ_3 -complete:*

- (1) $\{e \mid R_e \text{ is essentially } n\text{-bounded}\}$, where $n > 1$.
- (2) $\{e \mid R_e \text{ is essentially universal } n\text{-bounded}\}$, where $n > 1$.
- (3) $\{e \mid R_e \text{ is essentially bounded}\}$.

The proof is similar to that of Proposition 4.11. It is not clear whether $\{e \mid B_e^n \equiv B_\infty^n\}$ is Σ_3 -complete for all $n > 1$.

Next we further explore the notion of essential boundedness.

PROPOSITION 6.6. *Let $k \geq 3$. Let R be a k -bounded relation such that every equivalence class of R is non-trivial. Then R is essentially $\lfloor \frac{k}{2} \rfloor$ -bounded. In particular, if $k = 3$, then R is computable.*

PROOF. Let Γ be an enumeration of the pairs of distinct elements in R . Without loss of generality assume that at each stage Γ enumerates one pair. For each stage s , let D_s be the set of all elements appeared in some pair enumerated in Γ before stage s and let E_s be the equivalence relation on D_s generated by Γ before stage s . Then we assume also that $E_{s+1} \neq E_s$ for every s . By our assumption that no R -equivalence class is a singleton, $\bigcup_s D_s = \mathbb{N}$. We define a $\lfloor \frac{k}{2} \rfloor$ -bounded relation S and a reduction f from R to S simultaneously by stages.

At stage 0, let $S_0 = \omega$ and f_0 be empty. At stage $s + 1$, suppose we have already defined $\lfloor \frac{k}{2} \rfloor$ -bounded S_s on D_s with $S_s \subseteq E_s$ and f_s witnessing a reduction from E_s to S_s . Let (i, j) be the pair given by Γ at stage s . There are four cases.

1. $D_{s+1} = D_s \cup \{i, j\}$. In this case let S_{s+1} be the trivial extension of S_s to the domain D_{s+1} and let $f_{s+1} = f_s \cup \{(i, \min(i, j)), (j, \min(i, j))\}$.

2. $D_{s+1} = D_s \cup \{i\}$. Let C be the E_s -class of j . Define S_{s+1} be the trivial extension of S_s to the domain D_{s+1} and let $f_{s+1} = f_s \cup \{(i, \min f_s(C))\}$.
3. $D_{s+1} = D_s \cup \{j\}$. Similar to Case 2.
4. $D_{s+1} = D_s$. Let C and D be the E_s -classes of i and j , respectively. Then $C \cup D$ is an E_{s+1} -class. Define $S_{s+1} = S_s \cup \{(\min f_s(C), \min f_s(D))\}$ and let $f_{s+1} = f_s$.

It is easy to see by induction that for each S_s -class A , there is some E_s -class C such that $A \subseteq C$ and that the size of A is less than half of that of C . This guarantees that S is $\lfloor \frac{k}{2} \rfloor$ -bounded. ■

A corollary to the proof is that all 3-bounded relations are PC.

COROLLARY 6.7. *All 3-bounded relations are PC. In particular, $B_\infty^3 \leq H$.*

PROOF. Suppose R is 3-bounded. Use the same proof as above. The relation S constructed will be just ω , since Case 4 never happens. The only difference is that we do not know $\bigcup_s D_s = \mathbb{N}$ now, corresponding to the situation that f may be only partial computable. But then f witnesses the definition of R being PC. For the second assertion, just note that H is universal for PC relations. ■

This corollary describes a part of the general relationship between bounded relations and the halting equivalence and its jumps (see § 8). In particular, it is also true that $B_\infty^4 \not\leq H$. We postpone the proof to § 8, where a much more general result is stated and proved.

To consider another generalization of the above results, we define spectra for ceers.

DEFINITION 6.8. Let R be ceer. Then the *spectrum* of R is the set

$$\text{spec}(R) = \{n \in \mathbb{N} \mid n = |[x]_R| \text{ for some } x \in \mathbb{N}\}.$$

With this definition, Proposition 6.6 can be restated as follows: If R is k -bounded and $1 \notin \text{spec}(R)$, then R is essentially $\lfloor \frac{k}{2} \rfloor$ -bounded. The method above can be modified to show the following general results.

THEOREM 6.9. *Let $k > 1$ and R be a bounded relation.*

- (1) *If $\text{spec}(R) \subseteq [k, n]$ for some $n \geq k$, then R is essentially $\lfloor \frac{n}{k} \rfloor$ -bounded.*
- (2) *If $\text{spec}(R) \subseteq \{1\} \cup [k, 2k - 1]$, then R is PC.*

PROOF. The proof of (1) is a modification of the proof of Proposition 6.6, in which construction no action is taken until an equivalence class of size $\geq k$ is generated. The proof of (2) is the same as that of Corollary 6.7. ■

Nies [7] considered 2-bounded relations and FC relations from a different perspective.

DEFINITION 6.10. For equivalence relations R_1 and R_2 on \mathbb{N} , let $R_1 \vee R_2$ denote the *join* of R_1 and R_2 , which is the smallest equivalence relation containing both R_1 and R_2 .

Nies proved a Join Theorem which states that for any ceer R there are 2-bounded relations R_1 and R_2 such that $R = R_1 \vee R_2$ and $R_1 \cap R_2 = \omega$.

7. The saturation jump operator

Throughout this section we will consider equivalence relations defined on the set of all finite subsets of \mathbb{N} , or in usual notation, on $[\mathbb{N}]^{<\omega}$. There is a computable bijection between $[\mathbb{N}]^{<\omega}$ and \mathbb{N} , therefore we continue to regard the universe of our equivalence relations to be formally \mathbb{N} . However, it turns out to be less confusing to work with the space $[\mathbb{N}]^{<\omega}$ for the equivalence relations we are to define in this section.

DEFINITION 7.1. Let R be a ceer on \mathbb{N} and let $X \subseteq \mathbb{N}$. The R -saturation of X is defined by

$$[X]_R = \{y \in \mathbb{N} \mid \exists x \in X(xRy)\}.$$

DEFINITION 7.2. Let R be a ceer on \mathbb{N} . The *saturation jump* of R , denoted by R^+ , is an equivalence relation on $[\mathbb{N}]^{<\omega}$ defined by

$$xR^+y \iff [x]_R = [y]_R.$$

For $n \in \mathbb{N}$, the n -th *saturation jump* of R is inductively defined by

$$\begin{aligned} R^{0+} &= R, \\ R^{1+} &= R^+, \\ R^{(n+1)+} &= (R^{n+})^+. \end{aligned}$$

Again, formally the universe of each R^{n+} is \mathbb{N} , but it is also natural to indentify the elements as hereditarily finite sets of depth n . For example, the universe of R^{2+} could then be identified as the collection of finite sets of finite sets.

The reader should be warned about the choice of the word ‘‘jump’’ for the name of the operator. Although it is obvious that $R \leq R^+$ always holds, it is not true that for any ceer R , $R < R^+$. In fact, if R is computable, then so is R^+ . Below let us collect some basic facts about the saturation jump operator.

PROPOSITION 7.3. *Let E , E_1 and E_2 be arbitrary ceers and $n \in \mathbb{N}$.*

- (1) *If $E_1 \leq E_2$, then $E_1^{n+} \leq E_2^{n+}$.*
- (2) *If E is FC, then so is E^{n+} .*
- (3) *If E is CC, then so is E^{n+} .*
- (4) *If E has the property that every pair of distinct E -classes are computably separable, then so does E^{n+} .*
- (5) *E and E^{n+} as sets have the same Turing degree.*

PROOF. Without loss of generality assume $n = 1$. (1)–(3) and (5) are obvious. For (4), let $\neg xE^+y$ and assume without loss of generality that $a \in x$ is such that $\forall b \in y, \neg bEa$. For each $b \in y$, let A_b be a computable set separating $[b]_E$ from $[a]_E$. Let $A = \bigcup_{b \in y} A_b$. Then A is computable. It follows that the set $\{z \in [\mathbb{N}]^{<\omega} \mid \exists u \in z (u \notin A)\}$ is computable and it separates $[x]_{E^+}$ from $[y]_{E^+}$. ■

Next we consider essentially bounded relations and show that the saturation jump operator behaves properly on these relations.

THEOREM 7.4. *If E is essentially bounded and non-computable, then $E < E^+$.*

PROOF. Suppose B is k -bounded and f is computable witnessing that $E \leq B$. Assume g witnesses $E^+ \leq E$. For notational simplicity we first consider $k = 2$. We derive a contradiction by showing that E is computable. Given $x, y \in \mathbb{N}$, note that

$$\begin{aligned}
xEy &\Leftrightarrow \text{at least one pair of } \{x\}, \{y\}, \{x, y\} \text{ are } E^+\text{-equivalent} \\
&\Leftrightarrow \text{all of } \{x\}, \{y\}, \{x, y\} \text{ are } E^+\text{-equivalent} \\
&\Leftrightarrow \text{at least one pair of } g(\{x\}), g(\{y\}), g(\{x, y\}) \text{ are } E\text{-equivalent} \\
&\Leftrightarrow \text{all of } g(\{x\}), g(\{y\}), g(\{x, y\}) \text{ are } E\text{-equivalent} \\
&\Leftrightarrow \text{at least one pair of } f(g(\{x\})), f(g(\{y\})), f(g(\{x, y\})) \text{ are } B\text{-equivalent} \\
&\Leftrightarrow \text{all of } f(g(\{x\})), f(g(\{y\})), f(g(\{x, y\})) \text{ are } B\text{-equivalent} \\
&\Leftrightarrow \text{at least one pair of } f(g(\{x\})), f(g(\{y\})), f(g(\{x, y\})) \text{ are equal}
\end{aligned}$$

where the last condition is computable. For general k , note that, by the same argument as above,

$$\begin{aligned}
xEy &\Leftrightarrow \text{at least one pair of } g(\{x\}), g(\{y\}), g(\{x, y\}) \text{ are } E\text{-equivalent} \\
&\Leftrightarrow \text{all of } g(\{x\}), g(\{y\}), g(\{x, y\}) \text{ are } E\text{-equivalent} \\
&\quad (\text{letting } z_0 = g(\{x\}), z_1 = g(\{y\}), z_2 = g(\{x, y\}))
\end{aligned}$$

$$\begin{aligned}
 &\Leftrightarrow \text{at least one pair of } g(\{z_0\}), g(\{z_1\}), g(\{z_2\}), g(\{z_0, z_1\}), g(\{z_1, z_2\}), \\
 &\quad g(\{z_0, z_2\}), g(\{z_0, z_1, z_2\}) \text{ are } E\text{-equivalent} \\
 &\Leftrightarrow \text{all of } g(\{z_0\}), g(\{z_1\}), g(\{z_2\}), g(\{z_0, z_1\}), g(\{z_1, z_2\}), \\
 &\quad g(\{z_0, z_2\}), g(\{z_0, z_1, z_2\}) \text{ are } E\text{-equivalent} \\
 &\Leftrightarrow \dots\dots
 \end{aligned}$$

When the iteration on the right hand side produces more than k elements, say, w_0, \dots, w_m , where $m \geq k - 1$, we have that xEy iff at least one pair of the elements $f(w_0), \dots, f(w_m)$ are equal. This shows that E is computable, a contradiction. \blacksquare

Next we demonstrate that there are many ceers whose saturation jump is no more complicated. For this we define a general operation $R^{\omega+}$ for ceer R as follows. First define by induction on n a sequence of ceers E_n by

$$\begin{aligned}
 \langle x, 0 \rangle E_0 \langle y, 0 \rangle &\Leftrightarrow xRy \\
 \langle x, i \rangle E_{n+1} \langle y, i \rangle &\Leftrightarrow x = y \vee (i \leq n \wedge \langle x, i \rangle E_n \langle y, i \rangle) \vee (i = n + 1 \wedge xE_n^+ y)
 \end{aligned}$$

Eventually define $R^{\omega+}$ by

$$xR^{\omega+}y \iff \exists n(xE_n y).$$

It is then easy to see that $R^{n+} \leq R^{\omega+}$ for any $n \in \mathbb{N}$ and that the properties of Proposition 7.3 remain true for $n = \omega$.

PROPOSITION 7.5. *For any ceer R , $(R^{\omega+})^+ \equiv R^{\omega+}$. In particular, there are FC relations R such that $R^+ \equiv R$.*

PROOF. Let $E = R^{\omega+}$. For each x let k_x be the largest i such that $\langle y, i \rangle \in x$ for some y . Define a reduction function f of E^+ to E by $f(x) = \langle x, k_x + 1 \rangle$. This f works, since if uE^+v then $k_u = k_v$ and hence $f(u)E_{k_u+1}f(v)$. On the other hand if $\neg(uE^+v)$ then either $k_u \neq k_v$ or else $\neg(uE_{k_u}v)$, in either case $\neg(f(u)Ef(v))$. \blacksquare

The following fact shows that the saturation jump operation is not one-one on bi-reducibility degrees of ceers.

PROPOSITION 7.6. *There are ceers R_1 and R_2 such that $R_1 < R_2$ but $R_1^+ \equiv R_2^+$.*

PROOF. Let R_1 and R_2 be respectively defined by

$$\langle x, 0 \rangle R_1 \langle x, 1 \rangle \iff x \in K$$

and

$$\langle x, 0 \rangle R_2 \langle x, 1 \rangle R_2 \langle x, 2 \rangle \iff x \in K.$$

Then $R_1 \leq R_2$, R_1 is 2-bounded and R_2 is not essentially 2-bounded by the proof of Theorem 6.3, hence $R_2 \not\leq R_1$.

To see that $R_1^+ \equiv R_2^+$, it is enough to demonstrate a reduction from R_2^+ to R_1^+ . For this let g_1 and g_2 be two computable functions with the property that $g_1(x) \neq g_2(y)$, $\forall x, y$ and $x \in K \Leftrightarrow g_1(x) \in K \Leftrightarrow g_2(x) \in K$. Such functions are easy to construct. Now define a computable function f such that

$$\begin{aligned} f(\langle x, 0 \rangle) &= \{\langle g_1(x), 0 \rangle, \langle g_2(x), 0 \rangle\}, \\ f(\langle x, 1 \rangle) &= \{\langle g_1(x), 0 \rangle, \langle g_2(x), 1 \rangle\}, \text{ and} \\ f(\langle x, 2 \rangle) &= \{\langle g_1(x), 1 \rangle, \langle g_2(x), 1 \rangle\}. \end{aligned}$$

It is easy to see that f induces a required reduction. \blacksquare

Finally some curious examples of the effect of the saturation jump operator.

THEOREM 7.7. *Let H be the halting equivalence. Then $H < H^+$.*

PROOF. We first show that if $H^+ \leq H$, then H^+ is PC. Suppose a computable function f witnesses $H^+ \leq H$ and a partial computable function φ witnesses H is PC. We claim that $\varphi \circ f$ witnesses that H^+ is PC. For this it is enough to check that for any $x, y \in [\mathbb{N}]^{<\omega}$ with $\varphi(f(x)) \uparrow$ and $f(x) = f(y)$, we have $x = y$. By the definition of H , if $\varphi(f(x)) \uparrow$, then $[f(x)]_H$ is a singleton. Since f is a reduction, it follows that $[x]_{H^+}$ is computable. Therefore x , as a finite subset of \mathbb{N} , consists only of elements a with $\varphi_a(a) \uparrow$. It follows then $[x]_{H^+}$ is a singleton, thus $f(x) = f(y)$ implies that $x = y$.

Next we show that H^+ is not PC. Assume a partial computable function ψ witnesses that H^+ is PC. Then if $x \in [\mathbb{N}]^{<\omega}$ has more than one element in its H^+ -class, we have $\psi(x) \downarrow$. Now pick an arbitrary element $a \in K_0$, i.e., with $\varphi_a(a) \downarrow = 0$. Let $b = \psi(\{a\})$. Consider the function f defined by $f(i) = \psi(\{a, i\})$, for $i \in \mathbb{N}$. Then f is a total computable function. Moreover, $i \in K_0$ iff $f(i) = b$. Since K_0 is not computable, this is a contradiction. \blacksquare

THEOREM 7.8. *For any $n \in \mathbb{N}$, H^{n+} is not universal, i.e., $H^{n+} < R_\infty$.*

PROOF. Let R be a precomplete ceer. Then every pair of distinct R -equivalence classes are computably inseparable (see, e.g. [2]). We show that $R \not\leq H^{n+}$ for any $n \in \mathbb{N}$. For $n = 0$ this was done in Proposition 3.9. We assume $n \geq 1$. We use the union operation in the sense of set theory, i.e., for a set x , let

$$\bigcup x = \{z \mid \exists y \in x (z \in y)\}.$$

For $k \geq 1$, define inductively $\bigcup^{k+1} x = \bigcup(\bigcup^k x)$. Now fix $n \geq 1$ and assume that f is a computable function witnessing $R \leq H^{n+}$. For each $x \in \mathbb{N}$, let

$k(x) = \bigcup^{n-1} f(x) \cap \overline{K}$, then each $k(x)$ is a finite subset of \overline{K} . We claim that k is a constant function. Since otherwise, there are $x \neq y$ such that $k(x) \neq k(y)$. Without loss of generality assume $a \in k(x) \setminus k(y)$. Then $f(x)$ and $f(y)$ are not equivalent under H^{n+} , therefore $\neg xRy$. It follows that $[f(x)]_{H^{n+}}$ and $[f(y)]_{H^{n+}}$ are computably inseparable. But this is not so, since the computable set $\{z \mid a \in \bigcup^{n-1} z\}$ separates them, a contradiction.

Now suppose the function k takes the constant value $\{c_1, \dots, c_m\}$. Then for each $x \in \mathbb{N}$, $\bigcup^{n-1} f(x) \setminus \{c_1, \dots, c_m\}$ consists only of elements of K . It follows that there is a computable function g witnessing that $R \leq \omega$, a contradiction. ■

A similar argument, with a bit more notation, shows the same for $n = \omega$, but we omit it now since we will demonstrate a stronger result later.

THEOREM 7.9. *Let E be a finite dimensional ceer. Then for any $n \in \mathbb{N}$, $H \not\leq E^{n+}$.*

PROOF. This is a similar argument as in the preceding proof. Without loss of generality assume $E = R_{K_0, \dots, K_{m-1}}$ is universal m -dimensional and let $R = E^{n+}$. Assume toward a contradiction that $H \leq R$ via f . For each $x \in \mathbb{N}$, let $A_x = \bigcup^{n-1} f(x) \cap (K_0 \cup \dots \cup K_{m-1})$ and $B_x = \bigcup^{n-1} f(x) \setminus A_x$. We claim that B_x is a constant finite set for all $x \in K$. This is because, if $B_x \neq B_y$ for some $x \neq y \in K$, then $[f(x)]_R$ and $[f(y)]_R$ can be computably separated, hence $[x]_H$ and $[y]_H$ can be computably separated, which is false. Now it follows that there is an infinite set $S \subseteq K$ such that for any $x, y \in S$, $x \neq y$, $[A_x]_E \neq [A_y]_E$. But this is impossible since there are at most 2^m many distinct $[A]_E$ with $A \subseteq K_0 \cup \dots \cup K_{m-1}$. ■

THEOREM 7.10. *Let A be hypersimple. For any $n \in \mathbb{N}$, $\omega \not\leq F_A^{n+}$.*

PROOF. We have seen in Proposition 5.6 that the conclusion is true for $n = 0$. Suppose $n > 0$ and assume $\omega \leq F_A^{n+}$ via f . We derive a contradiction by defining a reduction g from ω to F_A . Let $g(0) = \sup \bigcup^{n-1} f(0)$. In general we will define each $g(m)$ as $\sup \bigcup^{n-1} \bigcup_{i \leq k} f(i)$ for some k . Suppose $g(m)$ is defined.

Let $s(z, n)$ be defined by the recursion $s(z, 0) = z$ and $s(z, n+1) = 2^{s(z, n)}$. Then $s(z, n)$ dominates the number of F_A^{n+} -equivalence classes in which there is an element x so that $\sup \bigcup^{n-1} x \leq z$.

Now let $g(m+1) = \sup \bigcup^{n-1} \bigcup_{i \leq s(g(m), n)+1} f(i)$. By the property of s mentioned above and the assumption on f , there is some element x with $g(m) < x \leq g(m+1)$ such that $\neg(g(m)F_A x)$. This implies that $\neg(g(m+1)F_A g(m))$. ■

8. The halting jump operator

In this section we introduce another jump operator which reminds us much of the halting jump in computability theory.

DEFINITION 8.1. Let E be an arbitrary ceer. The *halting jump* of E , denoted by E' , is the ceer defined by

$$xE'y \iff x = y \vee \varphi_x(x) \downarrow E\varphi_y(y) \downarrow$$

for $x, y \in \mathbb{N}$. For any $n \in \mathbb{N}$, the n -th *halting jump* of E , denoted by $E^{(n)}$, is inductively defined by

$$\begin{aligned} E^{(0)} &= E \\ E^{(1)} &= E' \\ E^{(n+1)} &= (E^{(n)})' \end{aligned}$$

DEFINITION 8.2. Let E be an arbitrary ceer. A ceer R is *partially classifiable by E* , denoted by PC^E , if there is a partial computable function ψ such that, for any $x, y \in \mathbb{N}$,

$$xRy \iff x = y \vee \psi(x) \downarrow E\psi(y) \downarrow .$$

Then we have the following basic properties.

PROPOSITION 8.3. *Let E be an arbitrary ceer.*

- (1) $E \leq E'$.
- (2) E' is universal for PC^E relations.
- (3) For $n \geq 1$, $id(n)' \equiv R_{K_0, \dots, K_{n-1}}$.
- (4) $\omega' = H$.
- (5) Let R be a ceer without computable classes. If $R \leq E'$ then $R \leq E$

PROOF. (1), (3) and (4) are obvious. (2) uses the same proof as that of Proposition 3.8. (5) is proved in the same way as Proposition 3.9. ■

Proposition 8.3 (5) implies that the halting jump of a non-universal ceer is non-universal. This feature of the halting jump operator is also a corollary of the following interesting result.

THEOREM 8.4. (The Halting Jump Theorem) *For ceers E_1 and E_2 , $E_1 \leq E_2$ iff $E_1' \leq E_2'$.*

PROOF. First suppose $E_1 \leq E_2$. Let f be a computable function such that $xE_1y \iff f(x)E_2f(y)$, for any $x, y \in \mathbb{N}$. First define a one-one computable function g so that for any $x, z \in \mathbb{N}$, $\varphi_{g(x)}(z) = f(\varphi_x(x))$. Then $\varphi_{g(x)}(g(x)) \downarrow$

iff $f(\varphi_x(x)) \downarrow$, and when this happens, they take the same value. Therefore, since g is one-one, we have

$$\begin{aligned} xE'_1y &\Leftrightarrow x = y \vee \varphi_x(x) \downarrow E_1\varphi_y(y) \downarrow \\ &\Leftrightarrow x = y \vee f(\varphi_x(x)) \downarrow E_2f(\varphi_y(y)) \downarrow \\ &\Leftrightarrow g(x) = g(y) \vee \varphi_{g(x)}(g(x)) \downarrow E_2\varphi_{g(y)}(g(y)) \downarrow \\ &\Leftrightarrow g(x)E'_2g(y) \end{aligned}$$

This shows that $E'_1 \leq E'_2$ via g .

Next suppose $E'_1 \leq E'_2$ via f . Then we have that for any $x \in K$, $f(x) \in K$. This is because, if $f(x) \notin K$, then $[f(x)]_{E'_2}$ is a singleton. Since f is a reduction, it follows that $[x]_{E'_1}$ is computable; but this can only happen when $x \notin K$ (if $x \in K$, then it is easy to see that $[x]_{E'_1}$ would be m -complete).

Now let us fix a one-one computable function s such that $\varphi_{s(x)}(s(x)) = x$. Then define a computable function g by $g(x) = \varphi_{f(s(x))}(f(s(x)))$. g is total since for any x , $s(x) \in K$ by definition, and therefore $f(s(x)) \in K$ by the claim in the preceding paragraph. Finally,

$$\begin{aligned} xE_1y &\Leftrightarrow s(x)E'_1s(y) \text{ (by definition of } s) \\ &\Leftrightarrow f(s(x))E'_2f(s(y)) \\ &\Leftrightarrow g(x)E_2g(y) \text{ (by definition of } E'_2 \text{ and } g) \end{aligned}$$

Therefore $E_1 \leq E_2$. ■

COROLLARY 8.5.

- (1) For ceers E_1 and E_2 , $E_1 \equiv E_2$ iff $E'_1 \equiv E'_2$.
- (2) If E is non-universal, then E' is non-universal.
- (3) For any ceer E , exactly one of the following holds:
 - i) for any $n \in \mathbb{N}$, $E^{(n)} \equiv E$;
 - ii) for any $n \in \mathbb{N}$, $E^{(n)} < E^{(n+1)}$.
- (4) For any $n \in \mathbb{N}$, $H^{(n)} < H^{(n+1)}$.
- (5) For any $n \in \mathbb{N}$, the sets $\{e \mid R_e \leq H^{(n)}\}$, $\{e \mid R_e \geq H^{(n)}\}$ and $\{e \mid R_e \equiv H^{(n)}\}$ are all Σ_3 -complete.
- (6) The set $\{e \mid R_e \text{ is PC}\}$ is Σ_3 -complete.

PROOF. (1) and (2) are immediate. For (3), note that $E \leq E' \leq \dots \leq E^{(n)} \leq E^{(n+1)} \leq \dots$. If ii) fails, then for some $n \in \mathbb{N}$, $E^{(n+1)} \leq E^{(n)}$; by applying Theorem 8.4 n times, we have $E' \leq E$, therefore $E \equiv E'$, which implies that $E^{(n)} \equiv E$ for any $n \in \mathbb{N}$. (4) follows immediately from (3) since $\omega < H = \omega'$. (5) follows immediately from Propositions 4.7–4.10. For (6) a straightforward computation shows that the set is Σ_3 . It is then

easy to check that if W_e is computable, then R_{W_e} is PC. The Halting jump theorem implies that if W_e is not computable, then $R_{W_e} \not\leq \omega' = H$, hence in particular it is not PC. Thus we have a reduction of Comp to $\{e \mid R_e \text{ is PC}\}$. ■

The next few theorems deal with bounded relations and their halting jumps.

THEOREM 8.6. *Let $n \in \mathbb{N}$ and R be an n -bounded relation. Then there is an $\lfloor \frac{n}{2} \rfloor$ -bounded relation S such that $R \leq S'$. Equivalently, $B_\infty^n \leq (B_\infty^{\lfloor \frac{n}{2} \rfloor})'$.*

PROOF. It is enough to define an $\lfloor \frac{n}{2} \rfloor$ -bounded relation S so that $R \in \text{PC}^S$, i.e., for some partial computable function ψ ,

$$xRy \iff x = y \vee \psi(x) \downarrow \wedge S\psi(y) \downarrow .$$

We fix an enumeration of the pairs of distinct elements in R and define S and ψ simultaneously. At the stage s of the enumeration let the pair (i, j) (say $i < j$) be enumerated for R and suppose S_s and ψ_s are defined. Consider four cases:

1. $i, j \notin \text{dom } \psi_s$. In this case extend ψ_s by letting $\psi_{s+1}(i) = \psi_{s+1}(j) = i$ and let $S_{s+1} = S_s$.
2. $i \in \text{dom } \psi_s$ but $j \notin \text{dom } \psi_s$. In this case extend ψ_s by letting $\psi_{s+1}(j) = \psi_s(i)$ and let $S_{s+1} = S_s$.
3. $i \notin \text{dom } \psi_s$ but $j \in \text{dom } \psi_s$. In this case similarly extend ψ_s by letting $\psi_{s+1}(i) = \psi_s(j)$ and let $S_{s+1} = S_s$.
4. $i, j \in \text{dom } \psi_s$. If in addition $\psi_s(i) = \psi_s(j)$ then do nothing; just let $\psi_{s+1} = \psi_s$ and $S_{s+1} = S_s$. Otherwise, $\psi_s(i) \neq \psi_s(j)$. Then let $\psi_{s+1} = \psi_s$ and $S_{s+1} = S_s \cup \{(\psi_s(i), \psi_s(j))\}$.

This finishes our construction of S and ψ . It follows easily from the construction that R is PC^S . Note that for any $x \in \text{dom } \psi$, $\psi(x)Rx$; in addition, there is $y \neq x$ with $\psi(y) = \psi(x)$. Also note that S admits new pairs only in Case 4 above, which implies that $S_s \subseteq R_s$ and that every S_s -class contains at most half many elements as the corresponding R_s -class. This implies that S is $\lfloor \frac{n}{2} \rfloor$ -bounded. ■

COROLLARY 8.7. *For any $n \in \mathbb{N}$ and $(2^{n+1} - 1)$ -bounded relation R , $R \leq \omega^{(n)}$. Equivalently, $B_\infty^{2^{n+1}-1} \leq \omega^{(n)}$.*

PROOF. By induction on n . The case $n = 0$ is obvious. Suppose R is $(2^{n+2} - 1)$ -bounded. Then by the previous theorem there is $(2^{n+1} - 1)$ -bounded relation S such that $R \leq S'$. By the inductive hypothesis $S \leq \omega^{(n)}$. Therefore $R \leq S' \leq \omega^{(n+1)}$. ■

It might be worth noticing that Proposition 6.6 is a corollary of Theorem 8.6, and that Corollary 6.7 is a special case of Corollary 8.7. In fact, if R is n -bounded and every equivalence class of R is non-trivial, then the proof of Theorem 8.6 produces a *total* function ψ witnessing the reduction from R to some $\lfloor \frac{n}{2} \rfloor$ -bounded ceer.

Next we show that the above estimates are in some sense optimal. In dealing with iterated halting jumps it is helpful to introduce the following notation. Let κ denote the partial computable function $\kappa(x) = \varphi_x(x)$, for any $x \in \mathbb{N}$, and for any $n \in \mathbb{N}$ let κ^n be the n -th iterate of κ . Then it is easy to see that, for any $x, y \in \mathbb{N}$,

$$x\omega^{(n)}y \iff \exists i \leq n(\kappa^i(x) \downarrow = \kappa^i(y) \downarrow).$$

We will use the following lemma.

LEMMA 8.8. *Let C be a computable subset of K . Then there is a total computable function $s(x)$ such that $\kappa(s(x)) \downarrow = x$ and $s(x) \notin C$, for any $x \in \mathbb{N}$.*

PROOF. For $i \in \mathbb{N}$ let $K_i = \{x \mid \kappa(x) \downarrow = i\}$. Then each K_i is non-computable. Note that for any $i \in \mathbb{N}$, $K_i \not\subseteq C$, since otherwise K_i would have a computable definition from the computability of C . This shows that for any $x \in \mathbb{N}$, the set $\{y \mid \kappa(y) \downarrow = x \wedge y \notin C\}$ is nonempty. Thus we may construct a total computable s so that for each x it returns an element from this set. \blacksquare

THEOREM 8.9. *For any $n \in \mathbb{N}$, $B_\infty^{2^{n+1}} \not\leq \omega^{(n)}$.*

PROOF. We define a sequence of bounded relations E_n by induction on n . Let E_0 be defined by

$$\langle x, 0 \rangle E_0 \langle x, 1 \rangle \iff \kappa(x) \downarrow.$$

Given E_n , define E_{n+1} by

$$\begin{aligned} \langle x, i \rangle E_{n+1} \langle x, j \rangle \iff & \langle x, i \rangle E_n \langle x, j \rangle \vee \langle x, i - 2^{n+1} \rangle E_n \langle x, j - 2^{n+1} \rangle \vee \\ & (i, j < 2^{n+2} \wedge \kappa^{n+2}(x) \downarrow). \end{aligned}$$

It is easy to see that each E_n is 2^{n+1} -bounded. We verify below that $E_n \not\leq \omega^{(n)}$ for any $n \in \mathbb{N}$, therefore establishing the theorem.

For $n = 0$ the conclusion is obvious. Now fix an arbitrary $n > 0$ and assume toward a contradiction that $E_n \leq \omega^{(n)}$ via some function f . We define by induction on $k \leq n$ computable sets C_k and computable functions s_k as follows. Let $C_0 = \{x \mid \exists l < 2^n (f(\langle x, 2l \rangle) = f(\langle x, 2l + 1 \rangle))\}$. Then C_0 is computable; also $C_0 \subseteq K$ since for any $x \in C_0$ there is some $l < 2^n$ such

that $\langle x, 2l \rangle E_n \langle x, 2l + 1 \rangle$, whereas if $x \notin K$ then $\neg(\langle x, i \rangle E_n \langle x, j \rangle)$ whenever $i \neq j$. Let s_0 be obtained from the preceding lemma for C_0 . Then for any $l < 2^n$ and any x , we have that $f(\langle s_0(x), 2l \rangle) \neq f(\langle s_0(x), 2l + 1 \rangle)$; but since $s_0(x) \in K$ and $\langle s_0(x), 2l \rangle E_n \langle s_0(x), 2l + 1 \rangle$ for any $l < 2^n$, it follows that for any $i < 2^{n+1}$, $\kappa(f(\langle s_0(x), i \rangle)) \downarrow$. This finishes the definition for the base step.

For the inductive step suppose we have defined computable set $C_k \subseteq K$ and computable function s_k for $0 \leq k < n$ with the following properties for any $x \in \mathbb{N}$:

- (a) $\kappa^{k+1}(s_k(x)) \downarrow = x$;
- (b) For any $i < 2^{n+1}$, $\kappa^{k+1}(f(\langle s_k(x), i \rangle)) \downarrow$.

Define C_{k+1} to be the set

$$\{x \mid \exists l < 2^{n-k-1} (f(\langle s_k(x), l2^{k+2} \rangle) \omega^{(k+1)} f(\langle s_k(x), l2^{k+2} + 2^{k+1} \rangle))\}.$$

It follows from property (b) that C_{k+1} is computable. To see that $C_{k+1} \subseteq K$, it is enough to show that for $x \in C_{k+1}$, $\kappa^{k+2}(s_k(x)) \downarrow$. Fix $x \in C_{k+1}$ and a witness $l < 2^{n-k-1}$. Note that for $i = l2^{k+2}$ and $j = l2^{k+2} + 2^{k+1}$, we have $f(\langle s_k(x), i \rangle) \omega^{(k+1)} f(\langle s_k(x), j \rangle)$ by the definition of C_{k+1} . It follows that $f(\langle s_k(x), i \rangle) \omega^{(n)} f(\langle s_k(x), j \rangle)$ and therefore $\langle s_k(x), i \rangle E_n \langle s_k(x), j \rangle$ since f witnesses $E_n \leq \omega^{(n)}$. But by the definition of E_n , it is easy to see that if $\langle y, i \rangle E_n \langle y, j \rangle$ and $j - i = 2^{k+1}$, then in fact $\langle y, i \rangle E_{k+1} \langle y, j \rangle$. Thus we actually have that $\langle s_k(x), i \rangle E_{k+1} \langle s_k(x), j \rangle$. On the other hand, by the definition of E_k and the fact that E_k is 2^{k+1} -bounded, $\neg(\langle s_k(x), i \rangle E_k \langle s_k(x), j \rangle)$ and $\neg(\langle s_k(x), i - 2^{k+1} \rangle E_k \langle s_k(x), j - 2^{k+1} \rangle)$. It follows from the definition of E_{k+1} that we must have $\kappa^{k+2}(s_k(x)) \downarrow$.

Let s' be obtained from the preceding lemma for C_{k+1} and let $s_{k+1}(x) = s_k(s'(x))$. We verify that properties (a) and (b) are preserved by this definition. (a) is immediate, since

$$\kappa^{k+2}(s_{k+1}(x)) \downarrow = \kappa(\kappa^{k+1}(s_k(s'(x)))) \downarrow = \kappa(s'(x)) \downarrow = x.$$

For (b), first note that for any $i < 2^{n+1}$, $\kappa^{k+1}(f(\langle s_{k+1}(x), i \rangle)) \downarrow$ by the definition of s_{k+1} . Thus if $k + 1 = n$ there is nothing to check. Suppose $k + 1 < n$. For any x , $s'(x) \notin C_{k+1}$, therefore, for any $l < 2^{n-k-1}$, $p = l2^{k+2}$ and $q = p + 2^{k+1}$, we have that $\neg(f(\langle s_{k+1}(x), p \rangle) \omega^{(k+1)} f(\langle s_{k+1}(x), q \rangle))$. By the inductive definition of E_{k+1} this implies that for any $i < 2^{n+1}$ there is $j < 2^{n+1}$, $j \neq i$, such that $\neg(f(\langle s_{k+1}(x), i \rangle) \omega^{(k+1)} f(\langle s_{k+1}(x), j \rangle))$. By property (a) for s_{k+1} , we know that $\langle s_{k+1}(x), i \rangle E_n \langle s_{k+1}(x), j \rangle$, for all $i, j < 2^{n+1}$. Therefore $f(\langle s_{k+1}(x), i \rangle) \omega^{(n)} f(\langle s_{k+1}(x), j \rangle)$, for all $i, j < 2^{n+1}$. Thus $\kappa^{k+2}(f(\langle s_{k+1}(x), i \rangle)) \downarrow$ for any $i < 2^{n+1}$.

In effect we have defined C_k, s_k which satisfy properties (a) and (b) for $k < n$, and C_n, s_n with property (a) only. We have that $\langle s_n(x), 0 \rangle E_n \langle s_n(x), 2^n \rangle$ by the property (a) for s_n . However, for any x , we also have $\neg(f(\langle s_n(x), 0 \rangle) \omega^{(n)} f(\langle s_n(x), 2^n \rangle))$. This contradicts the assumption that f is a reduction from E_n to $\omega^{(n)}$. ■

We have established in Corollary 8.7 that all bounded relations are reducible to some $\omega^{(n)}$. Next we show that FC relations are not necessarily so.

THEOREM 8.10. *Let A be hypersimple. Then for any $n \in \mathbb{N}$, $F_A \not\leq \omega^{(n)}$.*

PROOF. By induction on n . For $n = 0$ the conclusion is trivial. Suppose now $F_A \not\leq \omega^{(n)}$ but $F_A \leq \omega^{(n+1)}$ via some computable function f . Therefore for any $x, y \in \mathbb{N}$,

$$xF_Ay \iff \exists i \leq n+1 (\kappa^i(f(x)) \downarrow = \kappa^i(f(y)) \downarrow).$$

Note that there are infinitely many pairs (x, y) such that xF_Ay but $\kappa^n(f(x)) \downarrow \neq \kappa^n(f(y)) \downarrow$. This is because, if it were not the case, then one could modify f by changing its values at finitely many points so as to get that $F_A \leq \omega^{(n)}$, contradicting our hypothesis. Also note that the (infinite) set of all pairs with the above property is c.e. Let us fix a non-repeating computable enumeration of this set and denote its k -th element by (x_k, y_k) .

Now we define a computable function g as follows. Let $g(0) = x_0$. For $i > 0$ let $g(i)$ be the first element of the sequence $x_0, y_0, x_1, y_1, \dots$ such that for any $j < i$, $\kappa^{n+1}(f(g(j))) \neq \kappa^{n+1}(f(g(i)))$. Note that the definition makes sense because for each element z in the above sequence $\kappa^{n+1}(f(z)) \downarrow$, and because F_A is FC.

Thus g is a total computable function with the property that if $i \neq j$, then $\kappa^{n+1}(f(g(i)) \downarrow \neq \kappa^{n+1}(f(g(j)) \downarrow$, hence $\neg(g(i)F_Ag(j))$. This shows that $\omega \leq F_A$ via g , a contradiction to Proposition 5.6. ■

COROLLARY 8.11. *If A is hypersimple, then F_A is not essentially bounded.*

PROOF. This follows from the preceding theorem and Corollary 8.7. ■

Corollary 8.5 (6) can also be seen as an immediate corollary of Theorem 8.10 by using the theorem of Dekker we quoted earlier in the proof of Lemma 4.9.

From the alternative definition of $\omega^{(n)}$ using partial computable functions κ and its iterates we can define a natural limit of the $\omega^{(n)}$'s when n approaches infinity. Let $\omega^{(\omega)}$ be the ceer defined by

$$x\omega^{(\omega)}y \iff \exists i (\kappa^i(x) \downarrow = \kappa^i(y) \downarrow)$$

for any $x, y \in \mathbb{N}$. One may wonder if $F_A \leq \omega^{(\omega)}$ for hypersimple A . We show next that this is indeed the case. In fact, $\omega^{(\omega)}$ is universal for all ceers.

We need the following lemma.

LEMMA 8.12. *For any partial computable function ψ there is a one-one total computable function v such that*

$$\kappa(v(x)) = v(\psi(x)), \text{ for any } x \in \mathbb{N}.$$

Moreover, an index for v can be obtained effectively from an index of ψ .

PROOF. Let $s(e, i)$ be a one-one total computable function such that

$$\varphi_{s(e, \langle y, x \rangle)}(u) = \varphi_e(\varphi_y(x)).$$

Let $t(e)$ be a computable function such that $\varphi_{t(e)}(i) = s(e, i)$. By the fixed point theorem there is e_0 such that $\varphi_{t(e_0)} = \varphi_{e_0}$. Then

$$\varphi_{s(e_0, \langle y, x \rangle)}(u) = \varphi_{e_0}(\varphi_y(x)) = \varphi_{t(e_0)}(\varphi_y(x)) = s(e_0, \varphi_y(x)).$$

Let $r(y)$ be a computable function such that $\varphi_{r(y)}(x) = \langle y, \psi(x) \rangle$. By the fixed point theorem again, there is y_0 such that $\varphi_{r(y_0)} = \varphi_{y_0}$. Then $\varphi_{y_0}(x) = \varphi_{r(y_0)}(x) = \langle y_0, \psi(x) \rangle$. Therefore

$$\varphi_{s(e_0, \langle y_0, x \rangle)}(u) = s(e_0, \varphi_{y_0}(x)) = s(e_0, \langle y_0, \psi(x) \rangle).$$

Letting $v(x) = s(e_0, \langle y_0, x \rangle)$, it is easy to check that v is as required. ■

THEOREM 8.13. $\omega^{(\omega)}$ is universal for all ceers.

PROOF. Let R be an arbitrary ceer. Fixing a computable enumeration of R , let R_s be the equivalence relation on \mathbb{N} established by stage s of the enumeration, i.e., R_s is generated by the finitely many pairs enumerated up to stage s . Let p_i be the $i + 1$ -th prime number. Define a total computable function ψ by $\psi(p_i^s) = p_j^{s+1}$ if $j \leq i$ is the least element such that $jR_s i$, and $\psi(n) = 0$ for any n not a power of a prime. Let v be the one-one total computable function given by the preceding lemma. Let $f(i) = v(p_i)$. We claim that f is a reduction function from R to $\omega^{(\omega)}$.

It is easy to see that, for any $i, j \in \mathbb{N}$,

$$iRj \iff \exists s(\psi^s(p_i) \downarrow = \psi^s(p_j) \downarrow).$$

By induction on s one can also see that $\kappa^s(f(i)) = \kappa^s(v(p_i)) = v(\psi^s(p_i))$. Since v is one-one, we have that $\psi^s(p_i) \downarrow = \psi^s(p_j) \downarrow$ iff $\kappa^s(f(i)) \downarrow = \kappa^s(f(j)) \downarrow$. ■

We do not know if there are non-universal ceers R with $R' \equiv R$. But the following proposition suggests that it is probably unlikely.

PROPOSITION 8.14. *If $R' \equiv R$ then $\omega^{(n)} < R$ for any $n \in \mathbb{N}$.*

PROOF. Since $1 \leq R$ and $\omega < 1' = R_K$, we have that $\omega^{(n)} < 1^{(n+1)} \leq R^{(n+1)}$. But if $R' \equiv R$ then $R^{(n+1)} \equiv R$. Hence $\omega^{(n)} < R$. ■

9. Incomparability between the jump operators

In this section we show that the saturation jump operator and the halting jump operator produce incomparable ceers in a very strong sense. A typical instance is still the halting equivalence relation.

THEOREM 9.1. *For any $n \in \mathbb{N}$, $H^+ \not\leq H^{(n)}$.*

PROOF. For $n = 0$ this is the content of Theorem 7.7. Assume $n \geq 1$ and assume toward a contradiction $H^+ \leq H^{(n)}$ via f . Let $s(x)$ be a one-one computable function such that $\kappa(s(x)) \downarrow = x$ for all $x \in \mathbb{N}$. Let $h(x)$ be a one-one computable function such that $\kappa(h(x)) = \kappa(x) + n$ for all $x \in \mathbb{N}$. For $x = \{x_1, \dots, x_m\}$ define $g(x) = \{s(0), \dots, s(n-1), h(x_1), \dots, h(x_m)\}$. Then $xH^+y \Leftrightarrow g(x)H^+g(y) \Leftrightarrow f(g(x))H^{(n)}f(g(y))$ for any $x, y \in \mathbb{N}^{<\omega}$.

We claim that for any $x \in \mathbb{N}^{<\omega}$, $\kappa^n(f(g(x))) \downarrow$. Granting the claim, we would have $f(g(x))H^{(n)}f(g(y)) \Leftrightarrow \kappa^n(f(g(x)))H\kappa^n(f(g(y)))$, which implies that $H^+ \leq H$, a contradiction. To establish the claim, we prove a stronger statement that for $k \leq n-1$ and any $x \in \mathbb{N}^{<\omega}$, $\kappa^{k+1}(f(\{s(0), \dots, s(k)\} \cup x)) \downarrow$.

The proof of the statement is by induction on $k \leq n-1$. For $k = 0$ we need to see that $\kappa(f(\{s(0)\} \cup x)) \downarrow$ for all x . Assume not, there would be some x_0 such that $[f(\{s(0)\} \cup x_0)]_{H^{(n)}}$ is a singleton and therefore $[\{s(0)\} \cup x_0]_H$ is computable, which is impossible. In general suppose the statement is true for k and $k+1 \leq n-1$. Then for any x , $\kappa^{k+1}(f(\{s(0), \dots, s(k+1)\} \cup x)) \downarrow$ by the inductive hypothesis. Suppose for some x_0 $\kappa^{k+2}(f(\{s(0), \dots, s(k+1)\} \cup x_0)) \uparrow$. Without loss of generality assume $[x_0]_H \cap [\{s(0), \dots, s(k+1)\}]_H = \emptyset$. Then for any $y \in \mathbb{N}$,

$$\begin{aligned} \kappa(y) \downarrow = k+1 &\Leftrightarrow (\{s(0), \dots, s(k+1)\} \cup x_0) H^+ (\{s(0), \dots, s(k), y\} \cup x_0) \\ &\Leftrightarrow f(\{s(0), \dots, s(k+1)\} \cup x_0) H^{(n)} f(\{s(0), \dots, s(k), y\} \cup x_0) \\ &\Leftrightarrow \kappa^{k+1}(f(\{s(0), \dots, s(k+1)\} \cup x_0)) = \\ &\quad \kappa^{k+1}(f(\{s(0), \dots, s(k), y\} \cup x_0)) \end{aligned}$$

Since the computations on the last line are all convergent, this gives a decision procedure for $K_{k+1} = \{y \mid \kappa(y) \downarrow = k+1\}$, a contradiction. \blacksquare

On the other hand, it is also true that $H' \not\leq H^{n+}$ for any $n \in \mathbb{N}$. In fact we demonstrate a slightly stronger result.

THEOREM 9.2. *$H' \not\leq H^{\omega+}$.*

PROOF. Instead of considering the formal definition of $H^{\omega+}$ it is more convenient to view the universe of $H^{\omega+}$ as the set of all finite trees with terminal nodes labeled by natural numbers. For such a tree t denote its set of labels for terminal nodes by N_t .

Now the proof uses a similar argument as that of Theorem 7.8. Assume $H' \leq H^{\omega+}$ via f . Let $s(x)$ be a computable function such that $\kappa(s(x)) \downarrow = x$, for any $x \in \mathbb{N}$. Then for any $x, y \in \mathbb{N}$, $xHy \Leftrightarrow s(x)H's(y) \Leftrightarrow f(s(x))H^{\omega+}f(s(y))$. Now for any $x \in \mathbb{N}$, let $A_x = N_{f(s(x))} \cap K$ and $B_x = N_{f(s(x))} \setminus A_x$. We have that B_x is a constant finite set for all $x \in \mathbb{N}$, because otherwise it would follow that for some $x_0 \neq y_0$ the sets $[s(x_0)]_{H'}$ and $[s(y_0)]_{H'}$ are computably separable, which is impossible. It then follows that it is computable to decide whether $f(s(x))H^{\omega+}f(s(y))$, which in turn implies that H is computable, a contradiction. ■

COROLLARY 9.3. $H^{\omega+}$ is not universal for all ceers.

There is an interesting contrast between Corollary 9.3 and Theorem 8.13.

10. Open problems

To motivate further research let us mention the following problems which seem to be unsolved.

PROBLEM 10.1. *Is being universal a Σ_3 -complete property among ceers?*

PROBLEM 10.2. *Is there a non-universal ceer E with $E \equiv E'$?*

There are other, more technical problems we encountered in our research. But the above ones seem to be the most significant problems we know of. Also, in this paper we have been focusing on the general theory of reducibility among ceers, not paying too much attention to the applications of this theory. Most of the natural examples of ceers arise in algebra and logic. For example, the word problems on finitely presented algebraic structures are essentially ceers. The classical unsolvability results usually deal with a single equivalence class. New problems emerge when they are investigated as ceers and when the strong reducibility notion is involved. As an example let us mention the following problem.

PROBLEM 10.3. *Is the following statement true: for any ceer R there is a finitely presented semigroup whose word problem as a ceer is bi-reducible to R ?*

It is obvious that a similar statement for finitely presented groups is false, since if the word problem for a group is CC then it has to be computable.

Finally let us mention that the idea of considering reducibility between equivalence relations as a framework for the study of complexity of classification problems can also be applied elsewhere. For example, we could

consider feasibly computable reductions between computable equivalence relations and develop a parallel theory as did in this paper. Such a theory would probably shed light on structural complexity theory and provide deeper understanding of the open questions in the field.

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