Computability and the Symmetric Difference Operator

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Joint work with Uri Andrews, Steffen Lempp, Joe Miller and Noah Schweber
Inpsired by a mathoverflow question!

Outline

1. Motivation
2. Incompatible Δ degrees in $\mathscr{R}$
3. Condition C
4. Open Questions

Motivation

- Combinatorial operations are almost never well-defined on Turing degrees
  - For instance, given degrees $\mathbf{a}, \mathbf{b}$ we can choose representatives A, B so that $A \cap B = \emptyset$ or so that $A \cap B \in \mathbf{a} \vee \mathbf{b}$
- So the operation $\cap$ is *never* well-defined on (non-trivial) Turing degrees. But what about other basic set-theoretic operations such as Δ (symetric difference)?
- In this talk we'll investigate the class of degrees for which Δ is well-defined:
  - That is, the class of degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that if $A \in \mathbf{a}, B \in \mathbf{b}$ then $A \,\Delta\, B \in \mathbf{c}$.
  - Note that (by choosing $A \subset \{2x \mid x \in \omega\}, B \subset \{2x + 1 \mid x \in \omega\}$) we must have $\mathbf{c} = \mathbf{a} \vee \mathbf{b}$.

Easy Failure

Proposition

If X, Y, Z are independent Turing degrees then $A = X \oplus Y \oplus \varnothing$ and $B = X \oplus \varnothing \oplus Z$ are independent Turing degrees without a well-defined symetric difference.

Where the set S of Turing degrees is independent if
$$(\forall A \in S)\left(A \not\leq_{\mathbf{T}} \bigoplus S \setminus \{A\}\right).$$

Just definition chasing:

- Note that $A \Delta B = (X \Delta X) \oplus (Y \Delta \varnothing) \oplus (\varnothing \Delta Z) = \varnothing \oplus Y \oplus Z$.
- So $A \Delta B \not\geq_{\mathbf{T}} X$ hence $A \Delta B \not\equiv_{\mathbf{T}} A \oplus B \equiv_{\mathbf{T}} (A \oplus \varnothing) \Delta (\varnothing \oplus B)$.

Easy Success

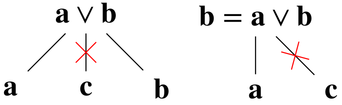
Proposition

If (for all $\mathbf{c}$*)*

$$(C) \qquad (\mathbf{c} \vee \mathbf{a} \geq \mathbf{b}) \wedge (\mathbf{c} \vee \mathbf{b} > \mathbf{a}) \implies \mathbf{c} \not< \mathbf{a} \vee \mathbf{b}$$

and $\mathbf{a} \neq \mathbf{b}$ *then* $\mathbf{a} \,\Delta\, \mathbf{b}$ *is well-defined.*

- So there is no degree $\mathbf{c} < \mathbf{a} \vee \mathbf{b}$ with:



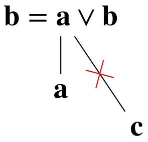
- But $A \,\Delta\, B \in \mathbf{c}$ for some such c so $A \,\Delta\, B \in \mathbf{a} \vee \mathbf{b}$

In $\mathscr{D}$

Proposition

There are degrees $\mathbf{a}, \mathbf{b}$ *for which* $\mathbf{a} \,\Delta\, \mathbf{b}$ *is well-defined.*

- Let $\mathbf{a}$ be a minimal degree and $\mathbf{b}$ be a strong minimal cover of $\mathbf{a}$ (any degree strictly below $\mathbf{b}$ is below $\mathbf{a}$).
- These degrees trivially satisfy C ;



- As any $\mathbf{c} < \mathbf{b}$ must actually satisfy $\mathbf{c} \leq \mathbf{a}$

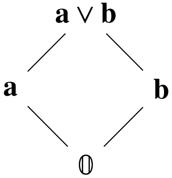
What about incompatible degrees?

Incompatible degrees in $\mathscr{D}$

Proposition

There are incompatible degrees $\mathbf{a}, \mathbf{b}$ *for which* $\mathbf{a} \,\Delta\, \mathbf{b}$ *is well-defined.*

- It is possible to embed the diamond as an initial segment of the Turing degrees[Sa63]



This satisfies (C). Remember:

$$(C) \qquad (\mathbf{c} \vee \mathbf{a} \geq \mathbf{b}) \wedge (\mathbf{c} \vee \mathbf{b} > \mathbf{a}) \implies \mathbf{c} \not< \mathbf{a} \vee \mathbf{b}$$

Working in $\mathscr{R}$

- So what about working in the r.e. degrees ($\mathscr{R}$)?

- Density means we can't hope to build the strong minimal covers we used above.

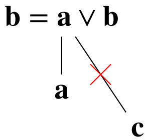
- But we can still satisfy C if none of the degrees below $\mathbf{a} \vee \mathbf{b}$ can join both $\mathbf{a}$ and $\mathbf{b}$ up to $\mathbf{a} \vee \mathbf{b}$.

Compatible Degrees in $\mathscr{R}$

Proposition

There are compatible degrees $\mathbf{a}, \mathbf{b}$ *in* $\mathscr{R}$ *for which* $\mathbf{a} \mathbin{\Delta} \mathbf{b}$ *is well-defined.*

- There is a pair of recursively enumerable degrees $\mathbf{a} < \mathbf{b}$ so that there is no Turing degree $\mathbf{c} < \mathbf{b}$ such that $\mathbf{a} \vee \mathbf{c} = \mathbf{b}$ [SS89, Co89].



- What about incompatible r.e. degrees?
- We'll investigate this case in the next section.

Outline

Incompatible degrees in $\mathscr{R}$

Theorem

There are (Turing) incomparable r.e. sets A and B such that for any $\widehat{A}$ and $\widehat{B}$ with $\widehat{A} \equiv_{\mathbf{T}} A$ and $\widehat{B} \equiv_{\mathbf{T}} B$, we have $\widehat{A} \,\Delta\, \widehat{B} \equiv_{\mathbf{T}} A \oplus B$.

Requirements

$\mathscr{P}_e^A$: $\Phi_e(A) \neq B$

$\mathscr{P}_e^B$: $\Phi_e(B) \neq A$

$\mathscr{R}_{i,j}$: $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B \implies \Gamma_{i,j}(\widehat{A}_i \,\Delta\, \widehat{B}_j) = A \oplus B$

Where:

$$\widehat{X}_{i,s}(z) = \begin{cases} \uparrow & \text{if } (\exists y < z)\widehat{X}_{i,s}(y)\uparrow \\ \Phi_{i,s}\left(X_s; z\right) & \text{otherwise} \end{cases}$$

$$\widehat{X}_i(z) = \lim_{s\to\infty} \widehat{X}_{i,s}(z)$$

Avoiding Index Profusion

Suppose there were e, e', k, k' with $\Phi_{e'}\left(\widehat{A}_e\right) = A, \Phi_{k'}\left(\widehat{B}_k\right) = B$ and $\widehat{A}_e \Delta \widehat{B}_k \not\equiv_{\mathbf{T}} A \oplus B$.

- Define i (likewise j replacing e, e' with k, k') so that:

$$\Phi_i(X;n) = \begin{cases} 1 - X(0) & \text{if } n = 0 \\ \Phi_e(X;n) & \text{if } n \neq 0 \wedge X(0) = A(0) \\ \Phi_{e'}(Y;n) & \text{if } n \neq 0, X(0) \neq A(0), \wedge \\ & \quad (\forall n)\,(Y(n) = X(n+1)) \end{cases}$$

- Then $\widehat{A}_i, \widehat{B}_j$ would witness failure with $\Phi_i\left(\widehat{A}_i\right) = A, \Phi_j\left(\widehat{B}_j\right) = B$.
  - As $\widehat{A}_i, \widehat{B}_j$ agree with $\widehat{A}_e, \widehat{B}_k$ except possibly at 0.
- Can use this anytime we have requirements on sets of same degree to avoid index profusion.

Meeting $\mathscr{P}_e^A, \mathscr{P}_e^B$

Requirement

$\mathscr{P}_e^A$: $\Phi_e(A) \neq B$

- We reserve some canidate (ball) x_e^A which we keep out of B unless we see $\Phi_e\left(A; x_e\right) = 0$.
- We place requirements on (downward growing) Π_2^0 tree and let possible canidates trickle down.
- Positive requirements grab and hold passing ball when they need a witness otherwise let canidates flow past to lower priority requirements.
  - This allows higher priority nodes to force canidates for lower priority nodes to be spaced out as needed.
  - Note that we throw away all balls when truepath moves to their left. This ensures that anytime a ball enters A or B all larger balls are discarded.

Framework for meeting $\mathscr{R}_{i,j}$

Requirement

$\mathscr{R}_{i,j}$: $\quad \Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B \implies \Gamma_{i,j}(\widehat{A}_i \,\Delta\, \widehat{B}_j) = A \oplus B$

- We conceptualize building $\Gamma_{i,j}$ via enumeration of axioms (always with large use).
- $\mathscr{R}_{i,j}$ works to build $\Gamma_{i,j}$ at stages where length of agreement for $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ increases.
- If x enters $A_s \oplus B_s$ we enumerate an axiom putting x into $\Gamma_{i,j}(\widehat{A}_i \,\Delta\, \widehat{B}_j)$ for all inputs.
- Enough to show that if $x \notin A \oplus B$ then we enumerate axiom saying so (i.e. $\Gamma_{i,j}(\widehat{A}_i \,\Delta\, \widehat{B}_j; x = 0)$)

Imagine we could hold $\widehat{B}_j$ fixed. How could we meet $\mathscr{R}_{i,j}$?

Hypothetically holding $\widehat{B}_j$ fixed

If $\widehat{B}_j$ was fixed then:

- If we see $\Phi_{i,s}\left(\widehat{A}_{i,s}; x\right) = 0$ (so $x \notin A$ if computation valid) then enumerate axiom saying $\Gamma_{i,j}(\widehat{A}_{i,s} \Delta \widehat{B}_{j,s}; 2x) = 0$ (e.g. guessing $A_s(x) = 0$).
- If later x enters A then $\widehat{A}_i$ and thus $\widehat{A}_i \Delta \widehat{B}_j$ must change below use (large at s) cancelling axiom.

DANGER! (As $\widehat{B}_j$ isn't fixed)

- Suppose at $s' > s, \hat{x}$ enters $\widehat{A}_i$ (restoring agreement between $\Phi_{i,s'}\left(\widehat{A}_{i,s'}\right)$ and $A_{s'}$)
- But then at $s'' > s', \hat{x}$ enters $\widehat{B}_j$ so $\widehat{A}_{i,s''} \Delta \widehat{B}_{j,s''}$ agrees with $\widehat{A}_{i,s} \Delta \widehat{B}_{j,s}$ on earlier use.

Hypothetically holding $\widehat{B}_j$ fixed

If $\widehat{B}_j$ was fixed then:

- If we see $\Phi_{i,s}\left(\widehat{A}_{i,s}; x\right) = 0$ (so $x \notin A$ if computation valid) then enumerate axiom saying $\Gamma_{i,j}(\widehat{A}_{i,s} \Delta \widehat{B}_{j,s}; 2x) = 0$ (e.g. guessing $A_s(x) = 0$).
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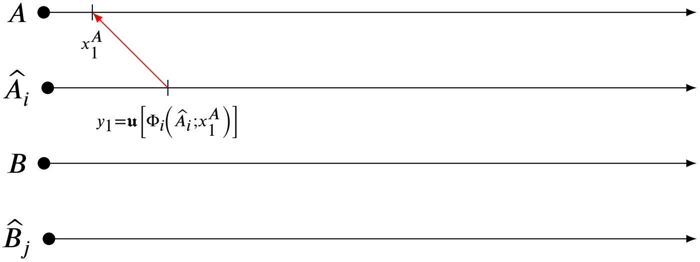
- But then at $s'' > s', \hat{x}$ enters $\widehat{B}_j$ so $\widehat{A}_{i,s''} \Delta \widehat{B}_{j,s''}$ agrees with $\widehat{A}_{i,s} \Delta \widehat{B}_{j,s}$ on earlier use.

Full Strategy

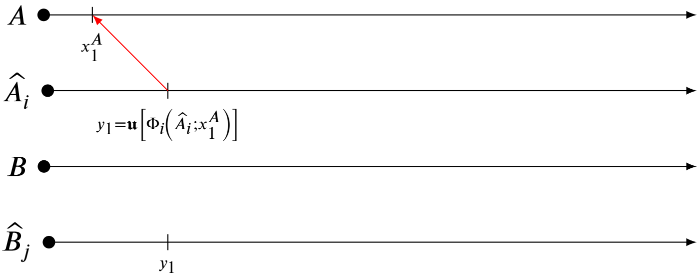
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Reserve x^A_1 to meet $\mathscr{P}^A_1$
- If x^A_1 enters A then $\widehat{A}_j$ changes below $y_1 = \mathfrak{u}\left[\Phi_i\left(\widehat{A}_i; x^A_1\right)\right]$

Full Strategy

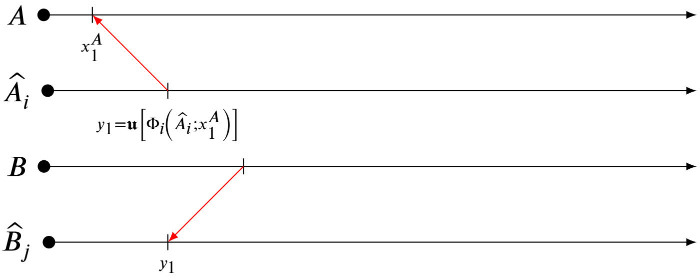
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x_k^A, x_k^B so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Want to preserve $\widehat{B}_j$ below $y_1 = \mathfrak{u}\left[\Phi_i\left(\widehat{A}_i; x_1^A\right)\right]$

Full Strategy

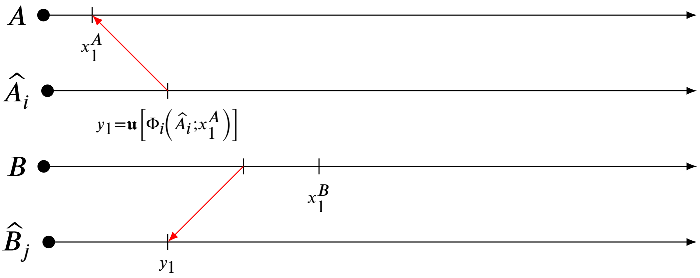
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Want to preserve $\widehat{B}_j$ below $y_1 = \mathfrak{u}\left[\Phi_i\left(\widehat{A}_i; x^A_1\right)\right]$

Full Strategy

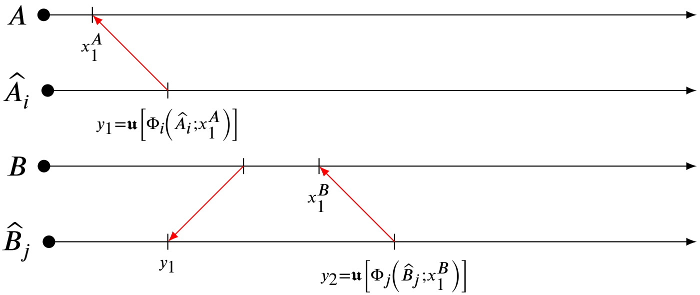
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x_k^A, x_k^B so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Want to preserve $\widehat{B}_j$ below $y_1 = \mathfrak{u}\left[\Phi_i\left(\widehat{A}_i; x_1^A\right)\right]$
- Preserve $\widehat{B}_j$ on $\mathfrak{u}\left[\Phi_j\left(B; y_1\right)\right]$ by picking $x_1^B > \mathfrak{u}\left[\Phi_j\left(B; y_1\right)\right]$

Full Strategy

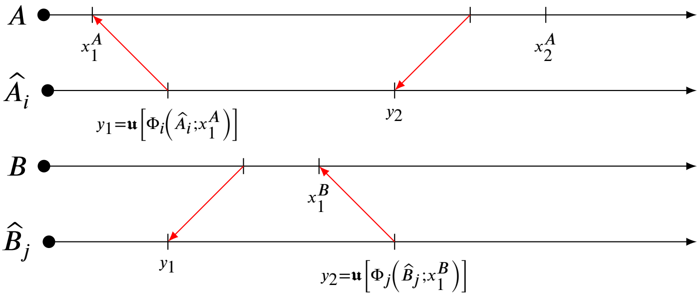
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Preserve $\widehat{B}_j$ on $\mathfrak{u}\left[\Phi_j\left(B; y_1\right)\right]$ by picking $x^B_1 > \mathfrak{u}\left[\Phi_j\left(B; y_1\right)\right]$
- Want to preserve $\widehat{A}_j$ below $y_2 = \mathfrak{u}\left[\Phi_j\left(\widehat{B}_j; x^B_1\right)\right]$

Full Strategy

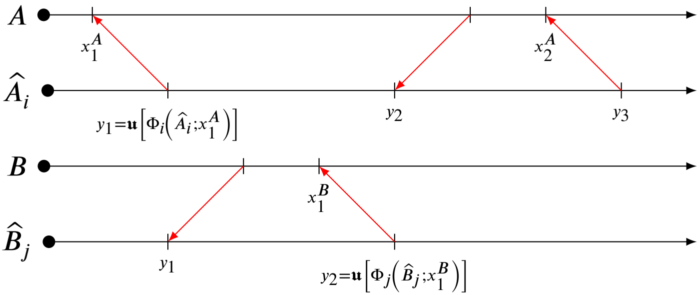
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x_k^A, x_k^B so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Want to preserve $\widehat{A}_j$ below $y_2 = \mathfrak{u}\left[\Phi_j\left(\widehat{B}_j; x_1^B\right)\right]$
- Preserve $\widehat{A}_j$ below y_2 by picking $x_2^A > \mathfrak{u}\left[\Phi_i\left(A; y_2\right)\right]$

Full Strategy

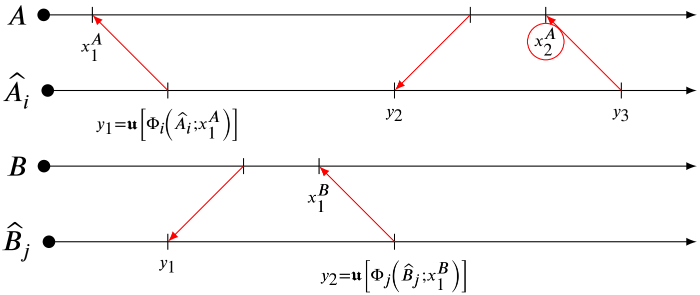
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Preserve $\widehat{A}_j$ below y_2 by picking $x^A_2 > \mathfrak{u}\left[\Phi_i\left(A; y_2\right)\right]$

Full Strategy

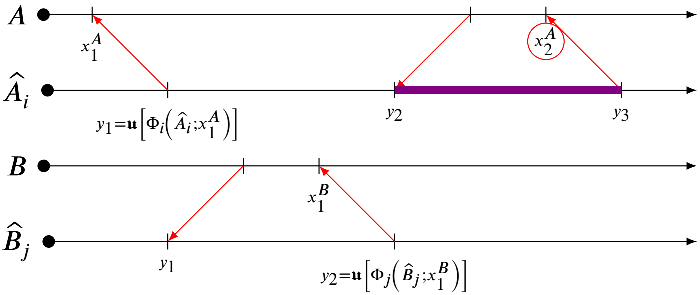
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Suppose x^A_2 enters A.

Full Strategy

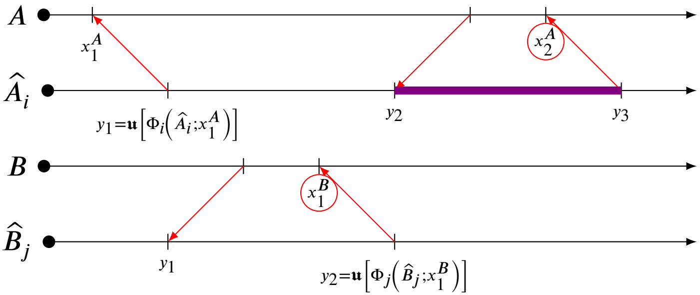
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Suppose x^A_2 enters A. Forces $\widehat{A}_i$ to change between y_2 and y_3

Full Strategy

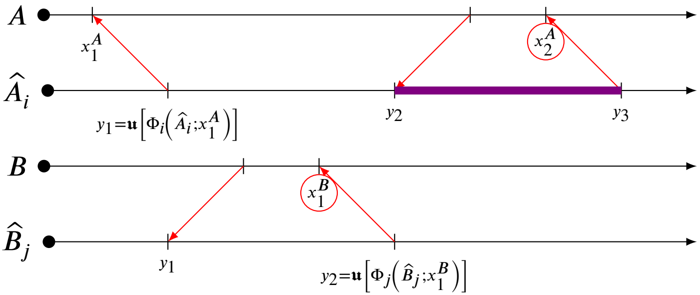
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Suppose x^A_2 enters A. Forces $\widehat{A}_i$ to change between y_2 and y_3
- There may *also* be changes above purple region but none below and there *must* be a change in it.

Full Strategy

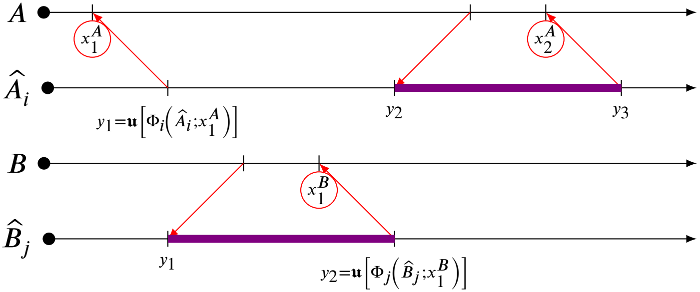
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Suppose x^A_2 enters A. Forces $\widehat{A}_i$ to change between y_2 and y_3
- Suppose x^B_1 enters B. Forces $\widehat{B}_j$ to change between y_1 and y_2

Full Strategy

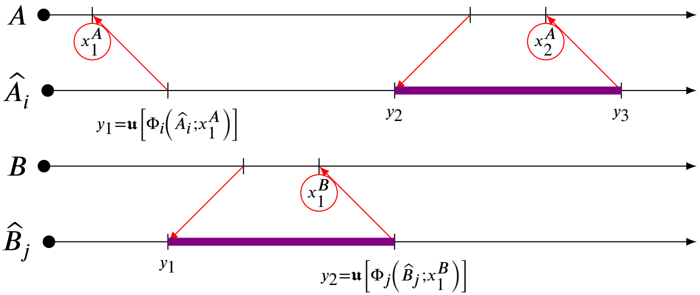
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Suppose x^B_1 enters B. Forces $\widehat{B}_j$ to change between y_1 and y_2

Full Strategy

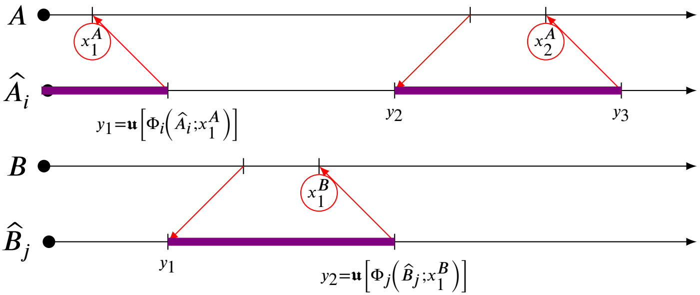
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Suppose x^B_1 enters B. Forces $\widehat{B}_j$ to change between y_1 and y_2
- Suppose x^A_1 enters A. Forces $\widehat{A}_i$ to change below y_1.

Full Strategy

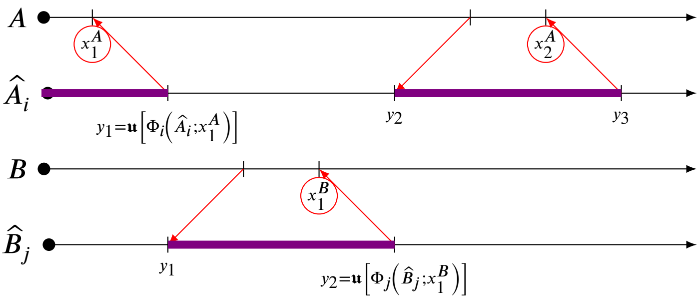
- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Suppose x^B_1 enters B. Forces $\widehat{B}_j$ to change between y_1 and y_2
- Suppose x^A_1 enters A. Forces $\widehat{A}_i$ to change below y_1.

Full Strategy

- Above outcome guessing $\Phi_i\left(\widehat{A}_i\right) = A \land \Phi_j\left(\widehat{B}_j\right) = B$ we space out witnesses x^A_k, x^B_k so only one of $\widehat{A}_i, \widehat{B}_j$ can change at a time.



- Note we never (below totality guess) allow both $\widehat{A}_i$ and $\widehat{B}_j$ to change at same location. So $\widehat{A}_i \,\Delta\, \widehat{B}_j$ never returns to prior value.
- When x^Z_k enters Z we pick new values for x^Y_m with $x^Y_m > x^Z_k$

Extending the Result

Theorem

There is a low, minimal pair of r.e. sets A and B such that for any $\widehat{A}$ and $\widehat{B}$ with $\widehat{A} \equiv_{\mathbf{T}} A$ and $\widehat{B} \equiv_{\mathbf{T}} B$, we have $\widehat{A} \,\Delta\, \widehat{B} \equiv_{\mathbf{T}} A \oplus B$.

That is we also ensure: $A' \equiv_{\mathbf{T}} B' \equiv_{\mathbf{T}} \mathbb{0}'$ and

$$(\forall X)\left(X \leq_{\mathbf{T}} A \wedge X \leq_{\mathbf{T}} B \implies X \leq_{\mathbf{T}} \mathbb{0}\right)$$

- Note that the usual minimal pair construction works by letting only one side (A or B) change at a time so is naturally compatible.
- Lowness only imposes finitary restraint so doesn't interfere.

Outline

Condition C

- Remember we started by looking at condition C:

$$(C) \qquad (\mathbf{c} \vee \mathbf{a} \geq \mathbf{b}) \wedge (\mathbf{c} \vee \mathbf{b} > \mathbf{a}) \implies \mathbf{c} \not< \mathbf{a} \vee \mathbf{b}$$

- But our construction above doesn't guarantee we produce A, B that satisfy C.
- We only proved that $\widehat{A}_i \,\Delta\, \widehat{B}_j \equiv_{\mathbf{T}} A \oplus B$ and thus isn't such a degree $\mathbf{c}$.
    - Might be possible way to build such a degree $\mathbf{c}$ which isn't of the form $\widehat{A}_i \,\Delta\, \widehat{B}_j$
    - So we haven't even shown that C is satisfied.
- Can we guarantee condition C is satisfied with incompatible r.e. degrees?
- If A, B are incompatible r.e. degrees with well-defined symetric difference must C be satisfied?

Condition C is satisfiable

Theorem

There are (Turing) incomparable r.e. sets A and B such that for any $C \leq_{\mathbf{T}} A \oplus B$ with $A \oplus C \geq_{\mathbf{T}} B$ and $B \oplus C \geq_{\mathbf{T}} A$, we have $C \equiv_{\mathbf{T}} A \oplus B$.

Requirements

$\mathscr{P}_e^A$: $\Phi_e(A) \neq B$

$\mathscr{P}_e^B$: $\Phi_e(B) \neq A$

$\mathscr{S}_{i,j,k}$: $\left(\Phi_i\left(A \oplus C_k\right) = B \wedge \Phi_j\left(B \oplus C_k\right) = A\right) \implies \Gamma_{i,j,k}(C_k) = A \oplus B$

Where $C_k = \Phi_k(A \oplus B)$

- We meet $\mathscr{P}_e^A$, $\mathscr{P}_e^B$ as before.

Meeting $\mathcal{S}_{i,j,k}$

Requirement

$\mathcal{S}_{i,j,k}$: $\quad \left(\Phi_i\left(A \oplus C_k\right) = B \wedge \Phi_j\left(B \oplus C_k\right) = A\right) \implies \Gamma_{i,j,k}(C_k) = A \oplus B$

- Think of C_k as playing the role of $\widehat{A}_i \,\Delta\, \widehat{B}_j$.
- If B is held fixed then a change in A forces a change in C_k (likewise for A, B switched) .
- Danger is that later change in B allows C_k to return to prior state (likewise for A).
- We use same spacing-out trick to ensure that changes to C_k as a result of an enumeration into A or B can't cancel each other out.
  - This ensures that an initial segment of C_k uniquely determines initial segment of A, B (assuming the antecedant is satisfied).

Condition C isn't necessary

Theorem

There are (Turing) incomparable r.e. sets A and B with a well-defined symetric difference and a set $C <_{\mathbf{T}} A \oplus B$ with $A \oplus C \geq_{\mathbf{T}} B, B \oplus C \geq_{\mathbf{T}} A$.

We build A, B, C and computations $\Xi(A \oplus B) = C$, $\Upsilon_1(A \oplus C) = B$, and $\Upsilon_2(B \oplus C) = A$ to satisfy:

Requirements

$\mathscr{P}_e^A$: $\Phi_e(A) \neq B$

$\mathscr{P}_e^B$: $\Phi_e(B) \neq A$

$\mathscr{R}_{i,j}$: $\Phi_i\left(\widehat{A}_i\right) = A \wedge \Phi_j\left(\widehat{B}_j\right) = B \implies \Gamma_{i,j}(\widehat{A}_i \,\Delta\, \widehat{B}_j) = A \oplus B$

$\mathscr{Q}_e$: $\Phi_i(C) \neq A \times B$

(Obviously, $A \times B \equiv_{\mathbf{T}} A \oplus B$)

Approach

- Use same strategy to meet $\mathscr{P}_e^X$. But how can we meet $\mathscr{R}_{i,j}$ without also meeting $\mathscr{S}_{i,j,k}$ (which ensured no such C existed)?
- **Goal**: make enumerations into A, B that ensure we see (and never reverse) a change in $\widehat{A}_i \Delta \widehat{B}_j$ but don't force us to change C.
- Note, computations using Ξ, Υ_k (those $\mathscr{S}_{i,j,k}$ breaks) depend on both A, B while computations in $\mathscr{R}_{i,j}$ between $A, \widehat{A}_i$ and $B, \widehat{B}_j$ only involve one of A, B.
- **Idea**: By freezing A and enumerating elements into B we can drive up use of $\Xi(A \oplus B)$ and $\Upsilon_2(B \oplus C)$ without affecting use of $\Phi_i(A)$ and vice versa.

Plan

- We find a pair x_k^A, x_k^B for enumeration into $A \times B$ such that:
  - Enumeration into $A \times B$ forces $\widehat{A}_i$ to change below any change in $\widehat{B}_j$ ensuring change in $\widehat{A}_i \,\Delta\, \widehat{B}_j$ (to meet $\mathscr{R}_{i,j}$).
  - But C is left unchanged by enumeration.
  - We find pair by reserving canidate for one side then enumerating elements into other side to push up uses and then vice versa.

- Hold x_k^A, x_k^B out of $A \times B$ until we see $\Phi_i(C; x_k^A, x_k^B)\downarrow = 0$ to meet $\mathscr{Q}_i$.

- Interleaved with these pairs we have the usual enumerations to meet $\mathscr{P}_e^X$ (keeping all canidates sufficently spaced out to meet $\mathscr{R}_{i,j}$)

Outline

Open Questions

Question

For what r.e. degrees $\mathbf{a}$ *does there exist an r.e. degree* $\mathbf{b}$ *with* $\mathbf{a} \Delta \mathbf{b}$ *well-defined.*

- Note that all our constructions have been compatible with the minimal pair construction.
- Raises tantalizing possibility that the class of r.e. degrees above is just the class of promptly simple degrees, aka, those part of a minimal pair.
- Possible easy disproof by tracking down the simple (compatible) examples cited up top and checking if they allow for non-promptly simple instaces.

More Open Questions

Question

For what r.e. degrees $\mathbf{a}$ *does there exist an incompatible r.e. degree* $\mathbf{b}$ *with* $\mathbf{a} \Delta \mathbf{b}$ *well-defined.*

- Maybe all examples are promptly simple except when $\mathbf{a} < \mathbf{b}$ (e.g. maybe you can stretch $\mathbf{b}$ up).
- Also would be interesting to ask the above questions but allow $\mathbf{b}$ to be any degree.
    - Perhaps one would want to start by looking at what r.e. degrees have been shown to have a strong minimal cover.

Question

Does every r.e. degree whose symetric difference is well-defined with respect to some degree have a well-defined symetric difference with respect to an r.e. degree? What if we restrict to incompatible degrees?

Other Directions

- Can one give a condition on $\mathbf{a}, \mathbf{b}$ which guarantees their symetric difference is well-defined. What about prevents?
- One might try and find a class of degrees such that any pair in it has a well-defined symetric difference.
    - Couldn't be very nice thanks to counterexample produced using 3 independent degrees.
- Do all examples of degrees with a well-defined symetric difference in some sense look like either the diamond or strong minimal cover cases or the r.e. examples?

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